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Unboundedness of a quadratically constrained convex quadratic function.

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UNBOUNDEDNESS OF A QUADRATICALLY CONSTRAINED CONVEX QUADRATIC FUNCTION

BY
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ABSTRACT

In this thesis we are concerned with the behaviour of convex quadratic functions constrained by convex and concave quadratic constraints. In particular, we are concerned with the unboundedness of the functions over feasible regions defined by the constraints.

For feasible regions defined only by convex constraints, or only by concave constraints, we present necessary and sufficient conditions for their unboundedness. We show that these conditions are equivalent to the existence of a feasible half line. We also present necessary and sufficient conditions for convex quadratic functions to be unbounded from above over the feasible regions, and to be unbounded from below over the feasible regions.

For feasible regions defined by both convex and concave constraints we present sufficient conditions for their unboundedness. Similarly, we present sufficient conditions for convex quadratic functions to be unbounded from above over the feasible regions, and to be unbounded from below over the feasible regions.

In all cases, we present numerical procedures for determining unboundedness. We also show that the implementation of the procedures only requires the solution of linear programmes having a very special structure.

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CHAPTER 1

INTRODUCTION

This thesis deals with certain aspects of the problem of minimizing a quadratic function subject to quadratic constraints (QPQC). The QPQC has been studied by many authors. In [20,21,22] Peterson and Ecker present a duality theory for the QPQC problem from a geometric programming point of view. Ecker and Niemi [9] and Fang and Rajasekera [10] use the results given by Peterson and Ecker to develop dual algorithms to solve the QPQC problem. Many other algorithms have also been proposed. In [6] Cole et al. give a reduced gradient method. in [1] Baron gives a cutting plane technique to solve the Lagrangian dual. in [23] Hao gives a triangularization method, and in [13] Mehrotra and Sun present a method of analytic centres.

The QPQC problem is significant because of its many applications. for example. to location and production planning problems [23]. and to nonlinear programming [27]. Other applications are given in Baron [1].

In this thesis we are concerned with the related problem of determining whether or not the convex quadratic function given by

$$Q(x) = a^T x + (1/2)x^T Bx.$$

where B is a real symmetric positive semidefinite (d, d)-matrix and a is a d-vector. is bounded from above and below on the following three feasible regions. We consider the quadratically constrained convex feasible region given by

$$\mathcal{R}^v = \{x \in \Re^d | Q_i(x) = a_i^T x + (1/2)x^T B_i x \leq c_i,\ i \in I_{m_1}\},$$

the quadratically constrained non-convex feasible region given by

$$\mathcal{R}^c = \{x \in \Re^d | Q_i(x) = a_i^T x - (1/2)x^T B_i x \le c_i,\ i \in I_{m\setminus m_1}\},$$

and the quadratically constrained non-convex feasible region given by

$$\mathcal{R} = \mathcal{R}^c \cap \mathcal{R}^r,$$

where, for each $i \in I_m$, B_i is a real symmetric positive semidefinite (d, d)-matrix, a_i is a d-vector, and c_i is a real scalar, and where $I_m = \{1, \ldots, m\}$, $I_{m_1} = \{1, \ldots, m_1\}$, and $I_{m\setminus m_1} = \{m_1 + 1, \ldots, m\}$. We assume that $\mathcal{R}^r$, $\mathcal{R}^c$ and $\mathcal{R}$ are not empty.

We present necessary and sufficient conditions for the objective to be unbounded from above over the feasible regions $\mathcal{R}^r$ and $\mathcal{R}^c$, and for the objective to be unbounded from below over the feasible regions $\mathcal{R}^r$ and $\mathcal{R}^c$. We also present sufficient conditions for the objective to be unbounded from below over the feasible region $\mathcal{R}$, and for the objective to be unbounded from above over the feasible region $\mathcal{R}$. Obviously, if a feasible region is bounded, then $Q(x)$ will be bounded from above and below over that feasible region. Therefore, we also present necessary and sufficient conditions for the boundedness of the regions $\mathcal{R}^r$ and $\mathcal{R}^c$, sufficient conditions for the boundedness of the region $\mathcal{R}$. In fact, we show that the regions $\mathcal{R}^r$ and $\mathcal{R}^c$ are unbounded if and only if they contain a half-line. By a "half-line" we mean the set of points which will be denoted by $x(p, s)$ and will be given by

$$x(p, s) = \{x \mid x = p + ts,\ t \ge 0\},$$

where s is nonzero.

In all cases, we present a numerical procedure for determining unboundedness. We note that this results in the first method for solving the following system of inequalities.

$$(1.1) \qquad s \neq 0, \; B_i s \neq 0 \text{ or } a_i^T s \leq 0, \; \forall i \in I_{m \setminus m_1}.$$

In order to implement the procedures, and, thus, in order to solve the particular system, it will be necessary to make use of the following lemma.

Lemma 1.1. *Let $P = [p_1, \ldots, p_d]$, and let $R' = \{s \mid a_i^T s \leq 0, \; \forall i \in I_m\}$. For $k = 1, \ldots, d$, define z_k^+ and z_k^- as follows:*

$$z_k^+ = \min\{+p_k^T s \mid s \in R'\}$$
$$z_k^- = \min\{-p_k^T s \mid s \in R'\}$$

Then $Ps = 0$ for all $s \in R'$ if and only if $z_k^+ = z_k^- = 0$, for each $k = 1, \ldots, d$.

∎

We note that the implementation of Lemma 1.1 only requires the solution of LPs where the constraints are all inequalities with zero right-hand-sides. These LPs have a single extreme point at the origin so that either the origin is a solution or else the LP is unbounded. We note, for interests sake, that solving such an LP is equivalent to determining whether or not there is a solution to System 1 of Farkas' theorem of the alternative [12].

Early work on these unboundedness problems can be found in [16-19]. We extend the results in those papers in that we drop the assumption that the rank of each of the Hessian matrices is equal to the number of nonzero diagonal elements, that is, that the submatrix formed by the nonzero columns and rows has full rank.

In the special case that all the constraint Hessian matrices are zero, then the feasible regions are linear. Thus, the results given here can be thought of as generalizations of both Corollary 3.4 in [14], which gives conditions for the unboundedness of linearly constrained feasible regions, and of the results in [7], which give the conditions for upper boundedness of a convex quadratic objective function over a linear feasible region.

We also generalize Theorem 8.4 in Rockafellar [24], which implies that $\mathcal{R}^r$ is unbounded if and only if it contains a half-line.

Necessary and sufficient conditions for the unboundedness of $Q(x)$ over $\mathcal{R}^r$ and for the unboundedness of $Q(x)$ over $\mathcal{R}^c$ can also be found in the unpublished research reports [4] and [5], respectively.

In Chapter 2 we present necessary and sufficient conditions for the boundedness of the regions $\mathcal{R}^r$, $\mathcal{R}^c$ and $\mathcal{R}$. In Chapter 3 we present necessary and sufficient conditions for the existence of upper and lower bounds on $Q(x)$ over the regions $\mathcal{R}^r$, $\mathcal{R}^c$ and $\mathcal{R}$. In Chapter 4 we present several results that, while they are of interest, did not fit well into the presentation of the main results. Finally, Chapter 5 contains the concluding remarks.

CHAPTER 2

UNBOUNDEDNESS OF THE FEASIBLE REGIONS

2.1. Introduction.

In Section 2.2 we consider the region $\mathcal{R}^v$, in Section 2.3 we consider the region $\mathcal{R}^c$, and in Section 2.4 we consider the region $\mathcal{R}$. We will show that, in each case, the region is unbounded if and only if it contains a half-line. We will then give necessary and sufficient conditions for the existence of the half-lines, and provide techniques for checking the conditions.

2.2 Unboundedness of $\mathcal{R}^v$.

In this section we will formulate the necessary and sufficient conditions for the convex region $\mathcal{R}^v$ to be unbounded.

Lemma 2.2.1. *The region $\mathcal{R}^v$ is unbounded if and only if it contains a half-line.*

Proof: See [24, Theorem 8.4].

∎

Lemma 2.2.2. *The nonempty feasible region $\mathcal{R}^v$ contains a half-line if and only if there exists a nonzero vector s satisfying*

(2.2.1) $$B_i s = 0, \quad \forall i \in I_{m_1}$$

(2.2.2) $$a_i^T s \le 0. \quad \forall i \in I_{m_1}.$$

Proof: The backward implication is proved first. Let p be any point in $\mathcal{R}^r$. For each $i \in I_{m_1}$ we have that $B_i s = 0$ so that

$$Q_i(p + ts) = a_i^T p + t a_i^T s + (1/2) p^T B_i p.$$

Condition (2.2.2.) and $p \in \mathcal{R}^r$ then imply that $Q_i(p + ts) \leq c_i$, $t \geq 0$. Thus $x(p, s) \in \mathcal{R}^r$. The forward implication is now proved. Let $p \in \mathcal{R}^r$ and let s be any n-vector. Suppose that for each $i \in I_{m_1}$ the inequality $Q_i(p+ts) - c_i \leq 0$ is satisfied for all $t \geq 0$. This implies that

$$[a_i^T p + (1/2) p^T B_i p - c_i] + [a_i^T s + p^T B_i s]t + [(1/2) s^T B_i s] t^2 \leq 0$$

is satisfied for all $t \geq 0$. If, for some k, $B_k s$ is nonzero, then, $s^T B_k s \neq 0$ since B_k is positive semidefinite. Thus, since $p \in \mathcal{R}^r$, $Q_k(p + ts) - c_i$ represents a convex parabola that is nonnegative for $t > N_1$, where N_1 is a finite positive root of $Q_k(p+ts) - c_i$. This contradiction proves that $B_i s = 0, \forall i \in I_{m_1}$, which is condition (2.2.1). Therefore the inequality reduces to

$$[a_i^T p + (1/2) p^T B_i p - c_i] + [a_i^T s]t \leq 0.$$

Clearly, this is satisfied for all $i \in I_{m_1}$ and for all $t \geq 0$ if and only if $a_i^T s \leq 0$, $\forall i \in I_{m_1}$, which is condition (2.2.2).

∎

Theorem 2.2.1. *The nonempty feasible region $\mathcal{R}^v$ is unbounded if and only if there exists a nonzero vector s satisfying (2.2.1) and (2.2.2).*

Proof: Follows from Lemmas 2.2.1 and 2.2.2.

∎

It is somewhat interesting to note that Theorem 2.2.1 is also valid for the feasible region

$$\mathcal{R}^{v<} = \{x \in \Re^d \mid Q_i(x) = a_i^T x + (1/2)x^T B_i x < c_i,\ i \in I_{m_1}\}.$$

To verify this, it is enough to show that if the region $\mathcal{R}^{v<}$ is nonempty then it is unbounded if and only if the region $\mathcal{R}^v$ is unbounded. This follows from the fact that if the region $\mathcal{R}^v$ contains an interior point p then any half-line $x(p,s)$ lying in $\mathcal{R}^v$ consists only of the interior points of this region.

Conditions (2.2.1)-(2.2.2) can be checked by first using Algorithm C, which is given in Appendix A, to make a change of variables in order to eliminate (2.2.1), and then using Lemma 1.1 to determine if the following reduced system

$$\underline{a}_i^T \underline{s} \leq 0,\ i \in I_{m_1}$$

has nonzero solution $\underline{s}$.

2.3. Unboundedness of $\mathcal{R}^c$.

This is perhaps the most important section in this thesis. It presents Algorithm A which can be used to determine whether or not there exists a solution to system (1.1). This algorithm is critical for dealing with the concave quadratic constraints.

We will show that $\mathcal{R}^c$ is unbounded if and only if it contains a half-line, and we will give necessary and sufficient conditions for the existence of a half-line in $\mathcal{R}^c$. In Lemmas 2.3.1 and 2.3.2 we give conditions under which there exists a vector s, to be used in Lemma 2.3.3, to obtain a half-line in the region $\mathcal{R}^c$.

The following notation will be used. Let Z be a subset of $I_{m\backslash m_1}$, and define $Z^c = \{i \in I_{m\backslash m_1} \mid i \notin Z\}$. The regions R_Z and R'_Z are given by

$$R_Z = \{x \in \Re^d \mid a_j^T x \leq c_j, \forall j \in Z\},$$

and

$$R'_Z = \{s \in \Re^d \mid a_j^T s \leq 0, \forall j \in Z\}.$$

Finally, the cone spanned by a set of vectors $s_i, i \in Z^c$, is given by

$$K_{Z^c} = \{x \in \Re^d \mid x = \sum_{i \in Z^c} \alpha_i s_i,\ \alpha_i \geq 0,\ \forall i \in Z^c\}.$$

Lemma 2.3.1. *Let Z^c will be arbitrary but fixed. If, for each $i \in Z^c$, there exists a vector s_i such that $s_i^T B_i s_i > 0$ then there exists a vector $s \in K_{Z^c}$ such that $s^T B_i s > 0, \forall i \in Z^c$.*

Proof: Without loss of generality, we can assume that $Z^c = \{m_1 + 1, \ldots, k\}$. The proof is by induction. For $k = m_1 + 1$, we set $s = s_{m_1+1}$. Suppose that a vector $\hat{s} \in K_{Z^c}$ satisfies $\hat{s}^T B_i \hat{s} > 0$, $\forall i \in \{m_1 + 1, \ldots, k - 1\}$. Now consider the point $\hat{s} + ts_k$. We have that

$$(\hat{s} + ts_k)^T B_i (\hat{s} + ts_k) = \hat{s}^T B_i \hat{s} + 2t\hat{s}^T B_i s_k + t^2 s_k^T B_i s_k. \tag{2.3.1}$$

For each $i \in \{m_1 + 1, \ldots, k - 1\}$ we have that either $s_k^T B_i s_k = 0$ or $s_k^T B_i s_k > 0$, and for $i = k$ we have that $s_k^T B_i s_k > 0$. If $s_k^T B_i s_k = 0$, then $B_i s_k = 0$ since B_i is positive semidefinite. Therefore, if $s_k^T B_i s_k = 0$, we have

$$(\hat{s} + ts_k)^T B_i (\hat{s} + ts_k) = \hat{s}^T B_i \hat{s} > 0. \ \forall t \geq 0.$$

If $s_k^T B_i s_k > 0$, then, since the expression in (2.3.1.) is nonnegative, it can be equal to zero only if the equality

$$(\hat{s}^T B_i s_k)^2 = (s_k^T B_i s_k)(\hat{s}^T B_i \hat{s}). \ i \in \{m_1 + 1, \ldots, k\}.$$

holds, that is, when

$$t = (-\hat{s}^T B_i s_k)/(s_k^T B_i s_k), \ \ i \in \{m_1 + 1, \ldots, k\}.$$

Then, if $(\hat{s}^T B_i s_k)^2 < (s_k^T B_i s_k)(\hat{s}^T B_i \hat{s})$ or $t \neq (-\hat{s}^T B_i s_k)/(s_k^T B_i s_k)$, $\forall i \in \{m_1 + 1, \ldots, k\}$, we have $(\hat{s} + t s_k)^T B_i (\hat{s} + t s_k) > 0$, $i \in \{m_1 + 1, \ldots, k\}$. We choose an arbitrary positive value $\hat{t}$ satisfying the above condition and set $s = \hat{s} + \hat{t} s_k$, noting that $s \in K_{Z^c}$.

∎

Lemma 2.3.2. *Let Z be arbitrary but fixed. For each $i \in Z^c$, there exists a vector $s_i \in R'_Z$ such that $s_i^T B_i s_i > 0$ and $s_i^T B_j s_i = 0$, $\forall j \in Z$, if and only if there exists a vector $s \in R'_Z$ such that $s^T B_i s > 0, \forall i \in Z^c$, and $s^T B_j s = 0, \forall j \in Z$.*

Proof: Suppose that for each $i \in Z^c$ there exists a vector $s_i \in R'_Z$ such that $s_i^T B_i s_i > 0$. From Lemma 2.3.1 it follows that there exists a vector $s \in K_{Z^c}$ such that $s^T B_i s > 0, \forall i \in Z^c$. Since $s \in K_{Z^c}$ and $s_i^T B_j s_i = 0, \forall i \in Z^c, \forall j \in Z$, then $s^T B_j s = 0, \forall j \in Z$. Also, since $s \in K_{Z^c}$ and $s_i \in R'_Z$, then for each $\forall j \in Z$ we have that

$$a_j^T s = \sum_{\forall i \in Z^c} \alpha_i a_j^T s_i \leq 0.$$

which implies that $s \in R'_Z$.

Now suppose that there exists an $s \in R'_Z$ such that $s^T B_i s > 0$, $\forall i \in Z^c$, and $s^T B_j s = 0$, $\forall j \in Z$. The result follows by setting $s_i = s$, $\forall i \in Z^c$.

∎

Lemma 2.3.3. *The region $\mathcal{R}^c$ contains a half-line $x(p, s)$ if and only if there exists a nonzero vector $s \in R'_{Z(s)}$, where $Z(s) = \{j \in I_{m \setminus m_1} | B_j s = 0\}$.*

Proof: Suppose that $x(p, s)$ is a half-line in $\mathcal{R}^c$ and that $s \notin R'_{Z(s)}$. Therefore, there exists an index $j \in Z(s)$ such that $a_j^T s > 0$. We have that

$$Q_j(p+ts) - c_j = Q_j(p) - c_j + ta_j^T s.$$

which is positive for t sufficiently large. This contradicts $x(p,s) \in \mathcal{R}^c$. Thus, $s \in R'_{Z(s)}$. Let $p \in \mathcal{R}^c$ and let $s \neq 0$ belong to $R'_{Z(s)}$. For $j \in Z(s)$, we have that

$$Q_j(p+ts) - c_j = Q_j(p) - c_j + ta_j^T s.$$

Since $a_j^T s \leq 0$, it follows that $Q_j(p+ts) - c_j \leq 0$ for all $t \geq 0$. For $i \in Z(s)^c$, we have that

$$Q_i(p+ts) - c_i = Q_i(p) - c_i + t\nabla Q_i(p)^T s - (t^2/2)s^T B_i s.$$

Since $s^T B_i s > 0$, we have that $Q_i(p+ts) - c_i \leq 0$ for t sufficiently large. Thus, there exists a scalar $T > 0$ such that $(p+ts) \in \mathcal{R}^c, \forall t > T$, and the half-line $x(\hat{p},s) \in \mathcal{R}^c$, where $\hat{p} = p + Ts$.

∎

Corollary 2.3.1 given below implies that it is only possible for $\mathcal{R}^c$ to be bounded if there are linear constraints. In fact, Corollary 2.3.2 shows that the linear constraints must provide the boundary of $\mathcal{R}^c$ in every direction s that is in the intersection of the range spaces of the nonzero constraint Hessian matrices.

Corollary 2.3.1. *If $B_{m_1+1}, \ldots, B_m$ are all nonzero, then $\mathcal{R}^c$ is unbounded.*

Proof: Clearly, for each B_i there exists a p_i such that $p_i^T B_i p_i > 0$. Set $Z = \emptyset$ so that $R'_{Z(s)} = \Re^d$ in Lemma 2.3.3. Then $\mathcal{R}^c$ contains a half-line and is therefore unbounded.

∎

Corollary 2.3.2. *If $\mathcal{R}^c$ is bounded, then for each nonzero*

$$s \in \bigcap_{\substack{i \in I_{m \setminus m_1} \\ B_i \neq 0}} R(B_i) \tag{2.3.2}$$

there exists an $i \in I_{m \setminus m_1}$ with $B_i = 0$ such that $a_i^T s > 0$.

Proof: We prove the contrapositive. Suppose that s satisfies (2.3.2). Let $p \in \mathcal{R}^c$ and consider the half-line $x(p, s)$. If $i \in I_{m \setminus m_1}$ with $B_i = 0$, then $a_i^T s \leq 0$ so that $Q_i(p+ts) = a_i^T(p+ts) \leq c_i$ for all $t \geq 0$. If $i \in I_{m \setminus m_1}$ with $B_i \neq 0$, then $s^T B_i s > 0$ so that $Q_i(p + ts) \leq c_i$ for all t sufficiently large. Thus, there exists a scalar $T > 0$ such that $(p + ts) \in \mathcal{R}^c, \forall t > T$, and the half-line $x(\hat{p}, s) \in \mathcal{R}^c$, where $\hat{p} = p + Ts$.

∎

The above results suggest the subsequent algorithm for determining the unboundedness of $\mathcal{R}^c$. Following the algorithm are some remarks concerning its implementation, and then some results concerning the algorithm's termination criteria.

ALGORITHM A: To determine the unboundedness of $\mathcal{R}^c$.

Step 1: Set $Z_0 = \{i \in I_{m\setminus m_1} | B_i = 0\}$. If $Z_0 = \emptyset$, then stop since $\mathcal{R}^c$ is unbounded. Otherwise, set $n = 0$ and go to Step 2.

Step 2: If there is a vector $s \neq 0$ in the region

$$R'_{Z_n} = \{s \in \Re^d \mid a_j^T s \leq 0, \forall j \in Z_n\}.$$

then go to Step 3. Otherwise, stop since $\mathcal{R}^c$ is bounded.

Step 3: Determine the region

$$\hat{Z}_n^c = \{i \mid i \in Z_n^c \text{ and } (B_i s = 0. \forall s \in R'_{Z_n})\}.$$

If $\hat{Z}_n^c = \emptyset$ then stop since $\mathcal{R}^c$ is unbounded. Otherwise, set $Z_{n+1} = Z_n \cup \hat{Z}_n^c$, replace n with $n+1$, and go to Step 2.

∎

We note that if $\mathcal{R}^c$ is unbounded, it follows from Algorithm A that $Z_n \subset Z$, where Z is given in Lemma 2.3.2. Also, it follows that $Z_n \subset Z(s)$, where $Z(s)$ is given in Lemma 2.3.3.

In Step 2 of the algorithm, it is necessary to determine whether or not there exists a nonzero vector $s \in R'_{Z_n}$. This can be done using Lemma 1.1, and it requires the solution of at most $2d$ LPs.

In Step 3, it is necessary to determine whether or not the set $\hat{Z}_n^c$ is empty. For each $i \in Z_n^c$ we must determine whether or not there is a vector $s \in R'_{Z_n}$ such that $B_i s \neq 0$. This can also be done using Lemma 1.1 with $P = B_i$. For each $i \in Z_n^c$ it is sufficient to solve at most $2d$ LPs

In fact, the number of LPs to be solved can, perhaps, be reduced. This is a direct consequence of the fact that, if in some iteration n, $(b_k)_i^T s$ or $-(b_k)_i^T s$ has an optimal value of zero, then it will be zero in all subsequent iterations. This is because $R'_{Z_{n-1}} \supseteq R'_{Z_n}$, for all n.

We will now, by way of Theorem 2.3.1 below, give a proof that Algorithm A terminates in a finite number of steps indicating whether or not $\mathcal{R}^c$ is bounded. However, we first show that if termination is in Step 2 then $\mathcal{R}^c$ is bounded (Lemma 2.3.4) and that if termination is in Step 3 then $\mathcal{R}^c$ is unbounded (Lemma 2.3.5). Following Theorem 2.3.1, we will prove, in Theorem 2.3.2, that $\mathcal{R}^c$ is unbounded if and only if it contains a half-line.

It is helpful in what follows to realize that Step 3 of Algorithm A implies that

(2.3.3) $$Z_n = Z_0 \cup \{ \bigcup_{\kappa=0}^{n-1} \hat{Z}_\kappa^c \}$$

(2.3.4) $$Z_0 \subset Z_1 \subset \cdots \subset Z_n$$

(2.3.5) $$R'_{Z_n} \subseteq R'_{Z_{n-1}} \subseteq \cdots \subseteq R'_{Z_0}.$$

(2.3.6) $$R_{Z_n} \subseteq R_{Z_{n-1}} \subseteq \cdots \subseteq R_{Z_0}.$$

The following notation and definitions will be used in Lemma 2.3.4. We define the sets $\hat{R}_{Z_\kappa}$, $\mathcal{R}^c_{Z_\kappa}$, $\kappa = 0, \ldots, n$, recursively, as follows:

ALGORITHM B.

Step 0: Define $M_j = 0, \forall j \in Z_0$ and set $\kappa = 0$.

Step 1: Set

$$\hat{R}_{Z_\kappa} = \{x \in \Re^d \mid a_j^T x \leq c_j + M_j,\ \forall j \in Z_\kappa\}. \tag{2.3.7}$$

$$\mathcal{R}^c_{Z_\kappa} = \{x \in \Re^d \mid Q_j(x) \leq c_j,\ \forall j \in Z_\kappa\}. \tag{2.3.8}$$

Step 2: If $\kappa = n$, then stop. Otherwise let $p_t^\kappa, t = 1, \ldots, \tau^\kappa$ be the set of all extreme points of $\hat{R}_{Z_\kappa}$. For each $j \in \hat{Z}_\kappa^c$ define

$$M_j = (1/2) \sum_{s=1}^{\tau^\kappa} \sum_{t=1}^{\tau^\kappa} (p_s^\kappa)^T B_j p_t^\kappa. \tag{2.3.9}$$

Replace κ with $\kappa + 1$ and go to Step 1.

∎

We note that since the sets R_{Z_κ}, $\kappa = 1, \ldots, n-1$ are unbounded, then so too are the sets $\hat{R}_{Z_\kappa}$, $\kappa = 1, \ldots, n-1$. Also, since R_{Z_n} is bounded so too is $\hat{R}_{Z_n}$. It is clear from the above algorithm, that

$$R_{Z_0} = \hat{R}_{Z_0} = \mathcal{R}^c_{Z_0} \tag{2.3.10}$$

and that

$$\mathcal{R}^c \subseteq \mathcal{R}^c_{Z_n} \subseteq \mathcal{R}^c_{Z_{n-1}} \cdots \subseteq \mathcal{R}^c_{Z_0}. \tag{2.3.11}$$

Lemma 2.3.4. *If Algorithm A terminates in Step 2, then $\mathcal{R}^c$ is bounded.*

Proof: If $\mathcal{R}^c_{Z_n} \subseteq \hat{R}_{Z_n}$, then it follows from (2.3.11) that

$$\mathcal{R}^c \subseteq \mathcal{R}^c_{Z_n} \subseteq \hat{R}_{Z_n}.$$

Since $\hat{R}_{Z_n}$ is bounded, this implies that $\mathcal{R}^c$ is bounded. We will prove, by induction, that $\mathcal{R}^c_{Z_n} \subseteq \hat{R}_{Z_n}$. It follows from (2.3.10) that $\mathcal{R}^c_{Z_0} \subseteq \hat{R}_{Z_0}$. Now suppose that $\mathcal{R}^c_{Z_\kappa} \subseteq \hat{R}_{Z_\kappa}$, for $\kappa = 0, \ldots, n-1$. Let $x \in \mathcal{R}^c_{Z_n}$. It then follows from (2.3.11) that $x \in \mathcal{R}^c_{Z_{n-1}}$. From the induction hypothesis, we then have that $x \in \hat{R}_{Z_{n-1}}$. Since $\hat{R}_{Z_{n-1}}$ is unbounded we can write $x = p + s$, where

$$p = \sum_{t=1}^{\tau^{n-1}} \lambda_t p_t^{n-1}, \quad \sum_{t=1}^{\tau^{n-1}} \lambda_t = 1, \quad \lambda_t \geq 0, \quad t = 1, \ldots, \tau^{n-1}, \quad s \in R'_{Z_{n-1}}.$$

It then follows from the definition of $\hat{Z}^c_{n-1}$ that $B_j s = 0$, $\forall\, j \in \hat{Z}^c_{n-1}$ and $s \in \mathcal{R}'_{Z_{n-1}}$. Therefore, we have that

$$\begin{aligned}
(1/2)x^T B_j x &= (1/2) \sum_{s=1}^{\tau^{n-1}} \sum_{t=1}^{\tau^{n-1}} \lambda_s \lambda_t (p_s^{n-1})^T B_j p_t^{n-1} \\
&\leq (1/2) \sum_{s=1}^{\tau^{n-1}} \sum_{t=1}^{\tau^{n-1}} (p_s^{n-1})^T B_j p_t^{n-1} \\
&= M_j, \quad \forall j \in \hat{Z}^c_{n-1}.
\end{aligned}$$

Therefore, if $Q_j(x) \leq c_j$, $j \in \hat{Z}^c_{n-1}$, then $a_j^T x \leq c_j + M_j$, $j \in \hat{Z}^c_{n-1}$. This combined with the fact that $x \in \hat{R}_{Z_{n-1}}$ shows that $x \in \hat{R}_{Z_n}$. Therefore, $\mathcal{R}^c_{Z_n} \subseteq \hat{R}_{Z_n}$.

∎

Lemma 2.3.5. *If Algorithm A terminates in Step 3, then $\mathcal{R}^c$ is unbounded.*

Proof: Since termination is in Step 3 then $\hat{Z}_n^c = \emptyset$. If $Z_n^c = \emptyset$ the result is obvious. Therefore, for each $i \in Z_n^c$ there exists a vector $s_i \in R'_{Z_n}$ such that $s_i^T B_i s_i > 0$. Let $j \in Z_n$ be arbitrary but fixed. It follows from (2.3.3) that there exists an index ℓ, $0 \leq \ell \leq (n-1)$, such that $j \in \hat{Z}_\ell^c$. Thus, $B_j s = 0$, $\forall s \in R'_{Z_\ell}$. Since $s_i \in R'_{Z_n}$, $\forall i \in Z_n^c$, it follows from (2.3.5) that $B_j s_i = 0$, $\forall j \in Z_\ell$, $\forall i \in Z_n^c$.

From Lemma 2.3.2, it now follows that there exists a nonzero vector $s \in R'_{Z_n}$ satisfying $s^T B_i s > 0$, $i \in Z_n^c$. Lemma 2.3.3 then implies the existence of a half-line in $\mathcal{R}^c$, which implies that $\mathcal{R}^c$ is unbounded.

∎

Theorem 2.3.1. *Algorithm A terminates after at most $m - m_1 - 1$ iterations indicating whether or not $\mathcal{R}^c$ is bounded.*

Proof: In the worst case, the algorithm will require $(m - m_1 - 1)$ iterations to terminate in Step 3 with $\hat{Z}_n^c$ being the empty set. If termination is in Step 1, then the Corollary 2.3.1 implies that $\mathcal{R}^c$ is unbounded. If termination is in Step 2, then Lemma 2.3.4 implies that $\mathcal{R}^c$ is bounded. If termination is in Step 3, then Lemma 2.3.5 implies that $\mathcal{R}^c$ is unbounded.

∎

Theorem 2.3.2. *The region $\mathcal{R}^c$ is unbounded if and only if it contains a half-line $x(p, s)$.*

Proof: If $\mathcal{R}^c$ contains a half-line, then it is clear that $\mathcal{R}^c$ is unbounded. Suppose that $\mathcal{R}^c$ is unbounded. It follows from Theorem 2.3.1, that Algorithm A would terminate in either Step 1 or Step 3. Suppose that termination is in Step 1. Then the proof of Corollary 2.3.1 implies the existence of a half-line in $\mathcal{R}^c$. Suppose that termination is in Step 3. The proof of Lemma 2.3.5 then implies the existence of a half-line in $\mathcal{R}^c$.

∎

The following examples demonstrate Algorithm A.

Example 1. Consider the feasible region in Figure 1(a), which is represented by the following constraint set.

$$
\begin{aligned}
x &\leq 2 \\
-x &\leq 0 \\
-y &\leq 0 \\
y - x^2 &\leq 1
\end{aligned}
$$

The algorithm proceeds as follows. In Step 1 we set $Z_0 = \{1,2,3\}$. In Step 2 we set $R_{Z_0} = \{(x,y)|x \leq 2, -x \leq 0, -y \leq 0\}$. We note, see Figure 1(b) that R_{Z_0} is unbounded. In Step 3 we set $R'_{Z_0} = \{(s_1, s_2)|s_1 \leq 0, -s_1 \leq 0, -s_2 \leq 0\}$. Clearly, $R'_{Z_0} = \{(s_1, s_2) = (0, s_2), s_2 \geq 0\}$. Since $B_4 x = 0$ for all $x \in R'_{Z_0}$, then $\hat{Z}_0^c = \{4\}$ and $Z_1 = \{1,2,3,4\}$. In Step 2 we set $R_{Z_1} = \{(x,y)|x \leq 2, -x \leq 0, -y \leq 0, y \leq 1\}$. Since R_{Z_1} is bounded (Figure 1(c)), then $\mathcal{R}^c$ is bounded. The proof of Lemma 2.3.4 states that since R_{Z_1} is bounded, then $\hat{R}_{Z_1}$ is bounded and that $\mathcal{R}^c \subseteq \hat{R}_{Z_1}$. To determine M_4, we note that the extreme points of $\hat{R}_{Z_0}$ are $(0,0)$ and $(2,0)$. Thus, $M_4 = 4$ and $\hat{R}_{Z_1}$, which is shown in Figure 1(d), is defined by the constraints

$$
\begin{aligned}
x &\leq 2 \\
-x &\leq 0 \\
-y &\leq 0 \\
y &\leq 5
\end{aligned}
$$

Example 2. Consider the feasible region in Figure 2, which is represented by the following constraint set.

$$
\begin{aligned}
x &\leq 2 \\
-x &\leq 0 \\
-y &\leq 0 \\
y &\leq 2 \\
y - x^2 &\leq 1
\end{aligned}
$$

The algorithm proceeds as follows. In Step 1 we set $Z_0 = \{1, 2, 3, 4\}$. In Step 2 we set $R_{Z_0} = \{(x, y) \mid x \leq 2, -x \leq 0, -y \leq 0, y \leq 2\}$. We note that R_{Z_0} is bounded. We stop since $\mathcal{R}^c$ is bounded.

Example 3. Consider the feasible region in Figure 3, which is represented by the following constraint set.

$$-x^2 - y \leq 1$$
$$-x^2 + y \leq 1$$
$$-x - (1/2)y^2 \leq 1$$

The algorithm proceeds as follows. In Step 1 we get that Z_0 is the empty set. We stop since $\mathcal{R}^c$ is unbounded

Example 4. Consider the feasible region in Figure 4, which is represented by the following constraint set.

$$-x^2 - y \leq 1$$
$$-x^2 + y \leq 1$$
$$-x - (1/2)y^2 \leq 1$$
$$x - (1/25)y^2 \leq 1$$

The algorithm proceeds as follows. In Step 1 we get that Z_0 is the empty set. We stop since $\mathcal{R}^c$ is unbounded.

Now we will present a lemma which puts the above results into a format that will be more useful in the subsequent sections.

Lemma 2.3.6. *The region $\mathcal{R}^c$ is unbounded if and only if there exists a vector s, satisfying system (1.1).*

Proof: Suppose that condition (1.1) is satisfied by a vector s, $||s|| \neq 0$. It then follows from Lemma 2.3.3 that $\mathcal{R}^c$ contains a half-line and is, therefore, unbounded. Now suppose that $\mathcal{R}^c$ is unbounded so that Algorithm A will terminate in either Step 1, or Step 3. If termination is in Step 1, then Lemma 2.3.1 implies the existence of a vector s satisfying $B_i s \neq 0, \forall i \in I_{m \setminus m_1}$. If termination is in step 3, then proof of Lemma 2.3.5 implies the existence of a vector s satisfying $B_j s = 0$ and $a_j^T s \leq 0$, for all $j \in Z_n$ and $B_i s \neq 0, \forall i \in Z_n^c$. In either case, condition (1.1) is satisfied.

∎

The next lemma gives the necessary and sufficient condition for $x(p,s)$ to be contained in $\mathcal{R}^c$.

Lemma 2.3.7. *The region $\mathcal{R}^c$ contains a half-line $x(p,s)$ if and only if $\forall i \in I_{m \setminus m_1}$ we have that*

(2.3.12)

$$a_i^T s \leq 0 \quad \vee$$
$$B_i s \neq 0 \wedge (-p^T B_i s + a_i^T s \leq 0 \vee (-p^T B_i s + a_i^T s)^2 \leq 2(c_i - Q_i(p)) s^T B_i s),$$

Proof: If condition (2.3.12) is satisfied then condition (1.1) is also satisfied and the region $\mathcal{R}^c$ contains the half-line. Now assume that the region $\mathcal{R}^c$ contains the half-line, $x(p,s)$. Then, from Lemma 2.3.6, we have that

$$a_i^T s \leq 0 \vee B_i s \neq 0, \forall i \in I_{m \setminus m_1}. \tag{2.3.13}$$

Assume that (2.3.12) is not satisfied. From (2.3.13) it follows that $\exists i \in I_{m\setminus m_1}$, such that

(2.3.14)

$$a_i^T s > 0 \wedge$$
$$[B_i s \neq 0 \wedge (-p^T B_i s + a_i^T s > 0 \wedge (-p^T B_i s + a_i^T s)^2 > 2(c_i - Q_i(p))s^T B_i s)]$$

Because $x(p, s) \in \mathcal{R}^c$, then

$$Q_i(p + ts) - c_i = Q_i(p) - c_i + t\nabla Q_i(p)^T s - (t^2/2)s^T B_i s \leq 0, \forall t \geq 0. \tag{2.3.15}$$

In the case when $s^T B_i s \neq 0$ these inequalities are satisfied only when the corresponding equations have at most one solution, i.e., when

$$(\nabla Q_i(p)^T)^2 + 2(Q_i(p) - c_i)(s^T B_i s) \leq 0.$$

This contradicts the inequality

$$(-p^T B_i s + a_i^T s)^2 > 2(c_i - Q_i(p))$$

in (2.3.14). Now assume that $s^T B_i s = 0$. Since $B_i s = 0$, then the inequality in (2.3.15) is equivalent to

$$Q_i(p) - c_i + a_i^T st \leq 0, \quad i \in I_{m \backslash m_1}. \quad \forall t \geq 0,$$

that is

$$t \leq (c_i - Q_i(p))/(a_i^T s).$$

Because $p \in \mathcal{R}^c$ and $a_i^T s > 0$, then the expression on the right-hand side of the last inequality is nonnegative, which contradicts the fact that the inequality should be satisfied $\forall t \geq 0$.

∎

2.4 Unboundedness of $\mathcal{R}$.

In this section we give sufficient conditions for unboundedness of the region $\mathcal{R}$. The results in this section are a simple combination of the results in the previous two sections.

Lemma 2.4.1. *The region $\mathcal{R}$ is unbounded if there exists a vector s, $||s|| \neq 0$, satisfying the following conditions*

$$\begin{aligned} &B_i s = 0, \ i \in I_{m_1}, \\ &a_i^T s \leq 0, \ i \in I_{m_1}, \\ &B_i s \neq 0 \ \vee \ a_i^T s \leq 0, \ i \in I_{m \backslash m_1}. \end{aligned} \tag{2.4.1}$$

Proof: Follows immediately from Theorem 2.2.1, Lemma 2.2.1, Theorem 2.3.2 and Lemma 2.3.6.

∎

We note that Algorithm A can be used to determine the existence of a solution to (2.4.1). Before using Algorithm A, we first make a change of variables to eliminate the equalities $B_i s = 0,\ i \in I_{m_1}$ (See Appendix A).

Also it should be noted, that conditions (2.4.1) are not necessary for the set R to be unbounded, since it can be unbounded without containing a half-line. Consider the simple example with feasible region in $\Re^2$ defined by the constraints $y - x^2 \leq 0$ and $y - x^2 \geq 0$.

The system equivalent to (2.4.1) can be written as

$$\underline{B}_i \underline{s} \neq 0 \ \vee \ \underline{a}_i^T \underline{s} \leq 0, \ i \in I_{m_1}, \tag{2.4.2}$$

where $\underline{B}_i = 0$, for $i \in I_{m_1}$. The following example demonstrates the procedure.

Example 5. Consider the feasible region in Figure 5, which is represented by the following constraint set

$$x^2 - y \leq -1$$
$$-x^2 - y \leq 1$$
$$x - (1/25)y^2 \leq 1.$$

We have, that $I_{m_1} = \{1\}$, $I_{m \backslash m_1} = \{2, 3\}$ and

$$B_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

so that $N_1 = (0,1)^T$. After eliminating the condition $B_1 s = 0$ using Algorithm C in Appendix A, we get

$$\begin{aligned} -\underline{s} &\leq 0 \\ 0\underline{s} &\neq 0 \vee -\underline{s} \leq 0 \\ (-2/25)\underline{s} &\neq 0 \vee 0\underline{s} \leq 0. \end{aligned}$$

We don't need to use Algorithm A to see that $\underline{s} = 1$ is a solution. It follows that $s = N\underline{s} = (0,1)^T$ satisfies conditions (2.4.1), so that $\mathcal{R}$ is unbounded.

CHAPTER 3

UNBOUNDEDNESS OF THE OBJECTIVE FUNCTION

3.1 Introduction.

In this chapter we present necessary and sufficient conditions for the existence of upper and lower bounds on $Q(x)$ over the three feasible regions $\mathcal{R}^r$, $\mathcal{R}^c$, and $\mathcal{R}$. $\mathcal{R}^v$. The following well known results are required.

Lemma 3.1.1. *The function $Q(x)$ is unbounded from below along a half-line $x(p, s)$ if and only if s satisfies*

$$Bs = 0, \tag{3.1.1}$$

$$a^T s < 0. \tag{3.1.2}$$

∎

Lemma 3.1.2. *The function $Q(x)$ is unbounded from above along a half-line $x(p, s)$ if and only if s satisfies either*

$$a^T s > 0 \quad \text{or} \, , \tag{3.1.3}$$

$$s^T Bs > 0 \quad (\text{i.e. } Bs \neq 0). \tag{3.1.4}$$

3.2 Unboundedness of $Q(x)$ on $\mathcal{R}^r$.

We first give the conditions for $Q(x)$ to be unbounded from below on $\mathcal{R}^r$, and afterwards we give the conditions for $Q(x)$ to be unbounded from above on $\mathcal{R}^r$.

Theorem 3.2.1. *The function $Q(x)$ is unbounded from below on $\mathcal{R}^r$ if and only if there is a vector s satisfying*

$$\begin{aligned} &Bs = 0, \\ &a^T s < 0, \\ &B_i s = 0, \quad i \in I_{m_1}, \\ &a_i^T s \leq 0, \quad i \in I_{m_1}. \end{aligned}$$

Proof: Follows directly from Lemmas 2.2.1, 2.2.2, 3.1.1 and Theorem 2.2.1.
∎

The conditions for the existence of a lower bound of $Q(x)$ on $\mathcal{R}^r$ can be checked by solving the following LP. It is interesting to note that the conditions are equivalent to System I of Slater's theorem of the alternative [12].

$$\text{(3.2.1)} \qquad \begin{aligned} \text{minimize } z &= a^T s \\ \text{subject to } Bs &= 0 \\ B_i s &= 0, \quad \forall i \in I_{m_1}, \\ a_i^T s &\leq 0, \quad \forall i \in I_{m_1}. \end{aligned}$$

We note that LP(3.2.1) is either unbounded from below or has the optimal objective value $z^* = 0$. If the LP is unbounded from below, then there exists a feasible s satisfying $a^T s < 0$ so that $Q(x)$ is unbounded from below on $\mathcal{R}^v$. Otherwise, $z^* = 0$, and $Q(x)$ is bounded from below on $\mathcal{R}^v$. As usual, we can use Algorithm C in Appendix A to eliminate the equality constraints and reduce the number of variables in the above LP.

Theorem 3.2.2. *The function $Q(x)$ is unbounded from above on $\mathcal{R}^v$ if and only if there is a vector s that satisfies*

$$
\begin{aligned}
&Bs \neq 0 \quad \vee \quad a^T s > 0, \\
&B_i s = 0, \ \forall i \in I_{m_1} \\
&a_i^T s \ \leq 0, \ \forall i \in I_{m_1}
\end{aligned}
$$

Proof: The result follows from Lemmas 2.2.1 and 3.2.2.
∎

The necessary and sufficient conditions for the upper bound of $Q(x)$ on $\mathcal{R}^v$ can also be checked by solving at most, $(2d+1)$ LPs, all having the same feasible region. As before, Algorithm C in Appendix A could be used to eliminate the equalities, thereby reducing the number of variables. To check the conditions, we first consider the following LP.

$$
\text{(3.2.2)} \qquad
\begin{aligned}
\text{maximize } z = \ & a^T s \\
\text{subject to } B_i s = 0, & \quad \forall i \in I_{m_1} \\
a_i^T s \leq 0, & \quad \forall i \in I_{m_1}.
\end{aligned}
$$

If the LP(3.2.2) is unbounded from above, then there exists a feasible s satisfying $a^T s > 0$, which implies that $Q(x)$ is unbounded from above on $\mathcal{R}^r$. Otherwise, $z^* = 0$ will be the optimal objective value so that no conclusion can be made. In that case we must determine if there is a vector s such that

$$
\begin{aligned}
&Bs \neq 0 \\
(3.2.3) \qquad &B_i s = 0, \quad \forall i \in I_{m_1} \\
&a_i^T s \leq 0, \quad \forall i \in I_{m_1}.
\end{aligned}
$$

This can be done using Lemma 1.1, after using Algorithm C in Appendix A to eliminate the equality constraints. We present four examples demonstrating the use of the reduced variable LPs. Note that the feasible region for each of the following examples is nonempty since all right-hand sides are nonnegative.

Example 6. The problem data is given as follows.

$$
B = \begin{pmatrix} 5 & 2 & 10 \\ 2 & 1 & 4 \\ 10 & 4 & 20 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 10 & 9 & 20 \\ 9 & 9 & 18 \\ 20 & 18 & 40 \end{pmatrix},
$$

$$
B_2 = \begin{pmatrix} 5 & 4.5 & 10 \\ 4.5 & 4.5 & 9 \\ 10 & 9 & 20 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 10 & 4 & 20 \\ 4 & 2 & 8 \\ 20 & 8 & 40 \end{pmatrix},
$$

$a = (1,1,1)^T$, $a_1 = (0,2,0)^T$, $a_2 = (1,1,0)^T$, $a_3 = (0,0,0)^T$, and $c_i = 100$, $i = 1,2,3$. We first determine whether or not $Q(x)$ is unbounded from below. Using

Algorithm C gives the change of variables $\underline{s} = N\underline{s}$, where $N = (-2,0,1)^T$, for LP(3.2.1). The resulting reduced variable LP is

$$\begin{aligned} \text{minimize } & -\underline{s} \\ \text{subject to } & 0\underline{s} \leq 0 \\ & -2\underline{s} \leq 0 \\ & 0\underline{s} \leq 0. \end{aligned}$$

Clearly, the LP is unbounded and $\underline{s} = 1$ is a feasible solution. This gives $s = (-2,0,1)$, which satisfies the conditions of Theorem 3.2.1, so that $Q(x)$ is unbounded from below.

We now determine whether or not $Q(x)$ is unbounded from above. The change of variables ($N = (-2,0,-1)^T$) gives the LP

$$\begin{aligned} \text{maximize } & -\underline{s} \\ \text{subject to } & 0\underline{s} \leq 0 \\ & -2\underline{s} \leq 0 \\ & 0\underline{s} \leq 0 \end{aligned}$$

which has solution $\underline{s} = 0$. We now consider the six LPs, arising from the use of Lemma 1.1. After the change of variables they have the same feasible region as above, that is, $\underline{s} \geq 0$, and optimal objective function values $z_j^+ = z_j^- = 0\underline{s}, j = 1,2,3$. Thus, there is no s satisfying the conditions of Theorem 3.2.2 and $Q(x)$ is bounded from above on $\mathcal{R}^v$.

Example 7. The problem data is given as follows.

$$B = \begin{pmatrix} 5 & 2 & 10 \\ 2 & 1 & 4 \\ 10 & 4 & 20 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 15 & 6 & 30 \\ 6 & 3 & 12 \\ 30 & 12 & 60 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 10 & 9 & 20 \\ 9 & 9 & 18 \\ 20 & 18 & 40 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 10 & 4 & 20 \\ 4 & 2 & 8 \\ 20 & 8 & 40 \end{pmatrix},$$

$c_0 = (0,1,1)^T$, $a_1 = (1,1,0)^T$, $a_2 = (0,2,0)^T$, $a_3 = (0,1,0)^T$, $c_1 = 0$, $c_2 = 10$, and $c_3 = 10$. We first determine whether or not $Q(x)$ is unbounded from below. Using Algorithm C - gives the change of variables $s = N\underline{s}$, where $N = (-2,0,1)^T$ for LP(3.2.1). The resulting reduced variable LP is

$$\begin{aligned} &\text{minimize } \underline{s} \\ &\text{subject to } -2\underline{s} \le 0 \\ &\qquad\qquad 0\underline{s} \le 0 \\ &\qquad\qquad 0\underline{s} \le 0. \end{aligned}$$

Clearly, this LP has optimal objective function value zero, so that $Q(x)$ is bounded from below. We now determine whether or not $Q(x)$ is unbounded from above. The change of variables ($N = (-2,0,1)^T$) gives the LP

$$\begin{aligned} &\text{maximize } \underline{s} \\ &\text{subject to } -2\underline{s} \le 0 \\ &\qquad\qquad 0\underline{s} \le 0 \\ &\qquad\qquad 0\underline{s} \le 0. \end{aligned}$$

This LP is unbounded and $\underline{s} = 1$ is a feasible solution. This gives $s = (-2,0,1)^T$, which satisfies the conditions of Theorem 3.2.2, so that $Q(x)$ is unbounded from below on $\mathcal{R}^v$.

Example 8. The problem data is given as follows.

$$B = \begin{pmatrix} 5 & 2 & 10 \\ 2 & 1 & 4 \\ 10 & 4 & 20 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 4 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 2 & 0 & -1 \\ 0 & .5 & 0 \\ -1 & 0 & .5 \end{pmatrix},$$

$a = (0,1,1)^T$, $a_1 = (1,0,-2)^T$, $a_2 = (-2,1,1)^T$, $c_1 = 0$, and $c_2 = 100$. We first determine whether or not $Q(x)$ is unbounded from below. Using Algorithm C gives that the intersection of the null spaces of B, B_1 and B_2 is the zero vector, which implies that there is no nonzero vector s satisfying (2.2.1), (2.2.2) and (3.1.1). Thus $Q(x)$ is bounded from below on $\mathcal{R}^v$.

We now determine whether or not $Q(x)$ is unbounded from above. The change of variables ($N = (1,0,2)^T$) gives the LP

$$\begin{aligned} &\text{maximize } 2\underline{s} \\ &\text{subject to } -3\underline{s} \leq 0 \\ &\qquad\qquad\quad 0\underline{s} \leq 0. \end{aligned}$$

This LP is unbounded and $\underline{s} = 1$ is a feasible solution. This gives $s = (1,0,2)^T$, which satisfies the conditions of Theorem 3.2.2 (clearly conditions (2.2.1), (2.2.2), 3.1.3)), which implies that $Q(x)$ is unbounded from above on $\mathcal{R}^v$.

Example 9. The problem data is given as follows.

$$B = \begin{pmatrix} 5 & 2 & 10 \\ 2 & 1 & 4 \\ 10 & 4 & 20 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 2.5 & 1 & 5 \\ 1 & .5 & 2 \\ 5 & 2 & 10 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 10 & 4 & 20 \\ 4 & 2 & 8 \\ 20 & 8 & 40 \end{pmatrix},$$

$a = (0,5,0)^T$, $a_1 = (0,1,0)^T$, $a_2 = (0,2,0)^T$, $c_1 = 0$, and $c_2 = 10$. Let us determine whether or not $Q(x)$ is unbounded from below. Using Algorithm C gives the change the variables $s = N\underline{s}$, where $N = (-2,0,1)^T$ for LP(3.2.1). The resulting reduced variable LP is

$$\begin{aligned} &\text{minimize } 0\underline{s} \\ &\text{subject to } 0\underline{s} \leq 0 \\ &\qquad\qquad\quad 0\underline{s} \leq 0 \end{aligned}$$

which has optimal objective function value zero so that $Q(x)$ is bounded from below on $\mathcal{R}^v$. We now determine whether or not $Q(x)$ is unbounded from above. The change of variables $(N = (-2,0,1))$ gives the LP

$$\begin{aligned} &\text{maximize } 0\underline{s} \\ &\text{subject to } 0\underline{s} \leq 0 \\ &\qquad\qquad\quad 0\underline{s} \leq 0 \end{aligned}$$

which has optimal solution $\underline{s} = 0$. Also all the six LPs arising from the use of Lemma 1.1, after the change of variables have the same feasible region as above, then optimal objective function values $z_j^+ = z_j^- = 0\underline{s}$, $j = 1,2,3$. Thus there is no s satisfying the conditions of Theorem 3.2.2 and $Q(x)$ is bounded also from above on $\mathcal{R}^v$.

3.3 Unboundedness of $Q(x)$ on $\mathcal{R}^c$.

In this section we will examine the unboundedness of the objective function $Q(x)$ over the feasible region $\mathcal{R}^c$.

The following theorems, which give necessary and sufficient conditions for the unboundedness of $Q(x)$ on $\mathcal{R}^c$ follow immediately from Lemmas 2.3.5, 3.1.1 and 3.1.2.

Theorem 3.3.1. *The function $Q(x)$ is unbounded from below on $\mathcal{R}^c$ if and only if there exists a vector s satisfying the following conditions.*

$$B s = 0 \tag{3.3.1}$$

$$a^T s < 0 \tag{3.3.2}$$

$$B_i s \neq 0 \text{ or } a_i^T s \leq 0, \ \forall i \in I_{m \backslash m_1}. \tag{3.3.3}$$

∎

In order to determine a nonzero s that satisfies conditions (3.3.1) - (3.3.3) we first make a change of variables to eliminate condition (3.3.1). Let N be a matrix whose columns are a basis for the null space of B. Note that if B has full rank then it is impossible to satisfy both (3.3.1) and (3.3.3) simultaneously so that $Q(x)$ is bounded from below on $\mathcal{R}^c$. We now set $s = N\underline{s}$, $\underline{a} = N^T a$, $\underline{a}_i = N^T a_i$, and $\underline{B}_i = N^T B_i N$ to get the equivalent system

$$\underline{a}^T \underline{s} < 0 \tag{3.3.4}$$

$$\underline{B}_i \underline{s} \neq 0 \text{ or } \underline{a}_i^T \underline{s} \leq 0, \ \forall i \in I_{m \backslash m_1}. \tag{3.3.5}$$

Let $\underline{a}_0 = \underline{a}$ and $\underline{B}_0 = 0$. Then any solution to conditions (3.3.4)-(3.3.5) is a solution to

$$\underline{B}_i \underline{s} \neq 0 \text{ or } \underline{a}_i^T \underline{s} \leq 0, \forall i \in \{0\} \cup I_{m \backslash m_1}. \tag{3.3.6}$$

We now observe that condition (3.3.6) is satisfied if and only if the region

$$\underline{\mathcal{R}}^c = \{\underline{x} \mid \underline{a}_i^T\underline{x} - (1/2)\underline{x}^T\underline{B}_i\underline{x} \le c_i,\ i \in \{0\} \cup I_{m\setminus m_1}\}$$

is unbounded, where c_0 is any constant so that $\underline{\mathcal{R}}^c$ is nonempty. We now apply Algorithm A to $\underline{\mathcal{R}}^c$ starting with $\underline{Z}_0 = \{0\} \cup Z_0$. If $\underline{\mathcal{R}}^c$ is bounded then there is no solution to condition (3.3.6), which implies that there is no solution to (3.3.4)-(3.3.5) so that $Q(x)$ is bounded from below on $\mathcal{R}^c$. If $\underline{\mathcal{R}}^c$ is unbounded, then, since $\underline{Z}_0 \ne \emptyset$, termination is in Step 3 of the algorithm with some set $\underline{Z} = \underline{Z}_n$. From Lemma 2.3.1, it follows that there exists a vector $\underline{s}^* \in R'_{\underline{Z}}$ where $\underline{B}_i\underline{s}^* \ne 0, i \in \underline{Z}^c$. We now consider the system

$$\text{(3.3.7)} \qquad \underline{a}^T\underline{s} < 0$$

$$\text{(3.3.8)} \qquad \underline{s} \in R'_{\underline{Z}}.$$

To determine whether or not there is a solution to conditions (3.3.7) and (3.3.8) we need only solve the LP $\min\{\underline{a}^T\underline{s} \mid \underline{s} \in R_{\underline{Z}'}\}$. If the LP has the solution $\underline{s} = 0$, then conditions (3.3.7)-(3.3.8) have no solution and $Q(x)$ is bounded from below on $\mathcal{R}^c$. Otherwise, the LP is unbounded from below and conditions (3.3.7)-(3.3.8) have a solution, say, $\hat{\underline{s}}$. Clearly, there then exists some $\lambda > 0$, such that

$$\text{(3.3.9)} \qquad \underline{a}_i^T(\underline{s}^* + \lambda\hat{\underline{s}}) < 0,$$

$$\text{(3.3.10)} \qquad (\underline{s}^* + \lambda\hat{\underline{s}}) \in R'_{\underline{Z}},$$

$$\text{(3.3.11)} \qquad \underline{B}_i(\underline{s}^* + \lambda\hat{\underline{s}}) \ne 0,\ \forall i \in \underline{Z}^c.$$

Conditions (3.3.9)-(3.3.11) imply that conditions (3.3.4)-(3.3.5) are satisfied, which imply that $Q(x)$ is unbounded from below on $\mathcal{R}^c$.

Theorem 3.3.2. *The function $Q(x)$ is unbounded from above on $\mathcal{R}^c$ if and only if there exists a vector s satisfying the following conditions.*

(3.3.12) $$Bs \neq 0 \text{ or } a^T s > 0$$

(3.3.13) $$B_i s \neq 0 \text{ or } a_i^T s \leq 0, \ \forall i \in I_{m \setminus m_1}$$

∎

Let $a_0 = -a$ and $B_0 = B$. Then any solution to conditions (3.3.12)-(3.3.13) is a solution to

(3.3.14) $$B_i s \neq 0 \text{ or } a_i^T s \leq 0, \ \forall i \in \{0\} \cup I_{m \setminus m_1}.$$

We now observe that condition (3.3.14) is satisfied if and only if the region

$$\underline{\mathcal{R}}^c = \{x \mid a_i^T x - (1/2) x^T B_i x \leq c_i, \ i \in \{0\} \cup I_{m \setminus m_1}\}$$

is unbounded, where c_0 is any constant so that $\underline{\mathcal{R}}^c$ is nonempty. We now apply Algorithm A to $\underline{\mathcal{R}}^c$. If $\underline{\mathcal{R}}^c$ is bounded then there is no solution to condition (3.3.14), which implies that $Q(x)$ is bounded from above on $\mathcal{R}^c$. If $\underline{\mathcal{R}}^c$ is unbounded, then the algorithm terminates with some set $\underline{Z} = \underline{Z}_n$. From Lemma 2.3.1, it follows that there exists a vector $s^* \in R'_{\underline{Z}}$ where $B_i s^* \neq 0, i \in \underline{Z}^c$. If $\{0\} \in \underline{Z}^c$, then s^* satisfies conditions (3.3.12)-(3.3.13) and $Q(x)$ is unbounded from above on $\mathcal{R}^c$. If $\{0\} \in \underline{Z}$, we now consider the system

$$a^T s > 0 \tag{3.3.15}$$

$$s \in R'_{\underline{Z}}. \tag{3.3.16}$$

To determine whether or not there is a solution to conditions (3.3.15) and (3.3.16) we need only solve the LP $\max\{a^T s \mid s \in R_{\underline{Z}'}\}$. If the LP has the solution $\underline{s} = 0$, then conditions (3.3.15)-(3.3.16) have no solution and $Q(x)$ is bounded from above on $\mathcal{R}^c$. Otherwise, the LP is unbounded from above and conditions (3.3.15)-(3.3.16) have a solution, say, $\hat{s}$. Clearly, there then exists some $\lambda > 0$, such that

$$a_i^T(s^* + \lambda\hat{s}) > 0, \tag{3.3.17}$$

$$(s^* + \lambda\hat{s}) \in R'_{\underline{Z}}, \tag{3.3.18}$$

$$B_i(s^* + \lambda\hat{s}) \neq 0, \ \forall i \in \underline{Z}^c. \tag{3.3.19}$$

Conditions (3.3.17)-(3.3.18) imply that conditions (3.3.12)-(3.3.13) are satisfied, which imply that $Q(x)$ is unbounded from above on $\mathcal{R}^c$.

It is interesting to note that if the function $Q(x)$ is bounded from above on the region $\mathcal{R}^c$, then the maximum value of $Q(x)$ on $\mathcal{R}^c$ is achieved at a vertex point of $\mathcal{R}^c$. This follows from a theorem proved by Ueing in [26] that if an arbitrary twice differentiable, convex objective function is bounded over a region defined by concave differentiable constraints then all local optimal solutions are at vertices of the feasible region.

Example 10. We consider the feasible region in Figure 3 which is represented by the constraint set in Example 3. We consider the function $Q(x) = -36x + y^2$. We first determine whether or not $Q(x)$ is unbounded from below. We have that

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

so that $N = (1,\ 0)^T$. After the change of variables, we get that conditions (3.3.4) and (3.3.5) reduce to $\underline{s} > 0$. Thus, $Q(x)$ is unbounded from below on $\mathcal{R}^c$. See Figure 6.

We now determine whether or not $Q(x)$ is unbounded from above. We consider the system in (3.3.14) and apply Algorithm A (using $c_0 = 1$). We stop in Step 1 with $Z_0 = \emptyset$. Using Lemma 2.3.1, we can construct the (nonunique) vector $\hat{s} = (1,\ 1)^T$ which satisfies $B_i\hat{s} \neq 0,\ i = 0, 1, 2, 3, 4$. Thus, $Q(x)$ is unbounded from above on $\mathcal{R}^c$. See Figure 6.

3.4 Unboundedness of $Q(x)$ on $\mathcal{R}$.

The results in this section combine the results in the previous two sections.

Theorem 3.4.1. *The function $Q(x)$ is unbounded from below on $\mathcal{R}$ if there exists a vector s, satisfying*

$$
(3.4.1) \qquad
\begin{aligned}
&Bs = 0,\\
&a^T s < 0,\\
&B_i s = 0,\ i \in I_{m_1},\\
&a_i^T s \le 0, i \in I_{m_1},\\
&B_i s \ne 0\ \vee\ a_i^T s \le 0,\ i \in I_{m \setminus m_1}.
\end{aligned}
$$

Proof: Proof follows from Lemmas 2.4.1 and 3.1.2.

∎

Theorem 3.4.2. *The function $Q(x)$ is unbounded from above on $\mathcal{R}$ if there is a vector s, that satisfies conditions*

$$
(3.4.2) \qquad
\begin{aligned}
&Bs \ne 0\ \vee\ a^T s > 0,\\
&B_i s = 0,\ i \in I_{m_1},\\
&a_i^T s \le 0,\ i \in I_{m_1},\\
&B_i s \ne 0\ \vee\ a_i^T s \le 0,\ i \in I_{m \setminus m_1}.
\end{aligned}
$$

Proof: Proof follows from Lemmas 2.4.1 and 3.1.1.

∎

To determine whether the systems (3.4.1) and (3.4.2) have a solution s, we can use a similar method to that used to determine whether or not the region $\mathcal{R}$ is unbounded. We first make a change of variables (see Algorithm C in Appendix A) to eliminate the conditions $B_i s = 0,\ i \in I_{m_1}$. This leaves the system

$$\underline{B}_i s \neq 0 \ \vee \ \underline{a}_i s \leq 0, \quad i \in \{0\} \cup I_{m_1}$$

where $\underline{B}_i = 0$.

Example 11. We consider the behaviour of the function $Q(x) = -36x + y^2$ over the unbounded feasible region in Figure 5, considered in Example 5. We first check the conditions (3.4.1), that is we first determine whether or not $Q(x)$ is unbounded from below. We have that

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } B_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

so that the intersection of the null spaces contains only the zero vector. Thus we cannot satisfy system (3.4.1) and we cannot determine if $Q(x)$ is bounded from below over the feasible region. However from Figure 7 it follows that $Q(x)$ is bounded from below over the feasible region. We now check the conditions of Theorem 3.4.2, that is, we determine whether or not $Q(x)$ is unbounded from above. The change of variable matrix is $N = (0,1)^T$, so that $\underline{s} \in \mathcal{R}$ and system (3.4.2) becomes the following.

$$\begin{aligned} \underline{s} &\neq 0 \\ 2\underline{s} &\neq 0 \vee 0\underline{s} \leq 0 \\ 0\underline{s} &\neq 0 \vee -s \leq 0 \\ 0\underline{s} &\neq 0 \vee -s \leq 0 \\ -(2/25)\underline{s} &\neq 0 \vee 0\underline{s} \leq 0. \end{aligned}$$

It is easy to note that $\underline{s} = 1$ is a solution. It follows that $s = N\underline{s} = (0, 1)^T$ satisfies conditions of the system (3.4.2) so that $Q(x)$ is unbounded from above on $\mathcal{R}$. See Figure 7.

CHAPTER 4

MISCELLANEOUS RESULTS

In this chapter we present results that are both interesting and related to the work in the previous chapters. However, they are included here rather than in the previous chapters so as not to interrupt the development of the main ideas.

The next two lemmas are concerned with determining if the region $\mathcal{R}^c$ is bounded in a coordinate direction. Let $R(B_i)$ denote the range space of the matrix B_i. We will use a superscript j to denote the j-th component of a vector.

Lemma 4.1. *If, for some j, there exists vector $s_i \in R(B_i)$ such that $s_i^j \neq 0$ then for some*

$$s \in \bigcap_{\substack{i \in I_{m \setminus m_1} \\ B_i \neq 0}} R(B_i), \quad s^j \neq 0.$$

Proof: The proof follows immediately from the construction of the vector s (see Lemma 2.3.1). If $s_i^j \neq 0$ for fixed i, j then the vector s_i can be considered as the last in the process of construction of the vector s. The value t corresponding to the vector s_i can be arbitrarily large (if positive) or arbitrarily small (if negative), but possibly different from the values

$$(-\hat{s}^T B_l s_i)/(s_i^T B_l s_i) \quad (if \;\; (\hat{s}^T B_l s_i)^2 = (s_i^T B_l s_i)(\hat{s}^T B_l \hat{s})).$$

∎

We note that if for some vector $s_i \in R(B_i)$, we have $s_i^j \neq 0$, for some j, then it follows from Corollary 2.3.2 and Lemma 4.9 that either there exists an $k \in I_{m \setminus m_1}$

with $B_k = 0$ such that $a_k^T s > 0$ or the region $\mathcal{R}^c$ is unbounded in the j-th coordinate direction.

Let S denotes any polyhedral region. It immediately follows from Lemma 4.1 that $s^j = 0$ for each nonzero

$$s \in \bigcap_{\substack{i \in I_{m \setminus m_1} \\ B_i \neq 0}} R(B_i) \cap S,$$

if and only if $s_i^j = 0$ for every $s_i \in R(B_i) \cap S$. The next Corollary follows from this observation. Let $Z = Z_n$, where Z_n is obtained from Algorithm A.

Lemma 4.2. *If, for some j, $s^j = 0$ for every nonzero*

$$s \in \bigcap_{\substack{i \in I_{m \setminus m_1} \\ B_i \neq 0}} R(B_i) \cap R'_Z,$$

then either $Z = I_{m \setminus m_1}$ or $\forall i \in I_d, \quad \exists s \in R'_Z, s^i = 0$, but not both.

Proof: From the statement given immediately before this Corollary, we have that if $s^j \neq 0$, then

$$s \in \bigcap_{i \in I_{m \setminus m_1}} N(B_i),$$

where $N(B_i)$ denotes null space of the matrix B_i.

If, for some j, each nonzero vector s belonging to R'_Z satisfies $s^j \neq 0$, then

$$\forall s \in R'_Z, \quad s \in \bigcap_{i \in I_{m \setminus m_1}} N(B_i),$$

which implies $Z = I_{m \setminus m_1}$. In the opposite case the second possibility holds, i.e. for each j, there exists a vector s belonging to R'_Z for which $s^j = 0$.

∎

In the next corollary we consider the set of constraints, which contains some concave quadratic functions with the same matrices B_i and vectors a_i as for the convex quadratic constraints.

Lemma 4.3. *The region $\mathcal{R}^v$ is unbounded if and only if the region $\mathcal{R}^{v'}$ is unbounded, where $\mathcal{R}^{v'}$ is defined by the constraint set obtained by adding the constraints*

$$a_i^T x - (1/2)x^T B_i x \leq c_i, \quad i = m_1 + 1, \; \dots \; , \; m + k,$$

where $k \leq m_1$, $a_{m_1+j} = a_j$, $B_{m_1+j} = B_j$, and $c_{m_1+j} \geq c_j$, for $j = m_1 + 1, ..., k$, to the constraints defining $\mathcal{R}^v$.

Proof: If $\mathcal{R}^v$ is unbounded, then Lemma 2.2.2 implies that (2.2.1) and (2.2.2) are satisfied. Thus, $B_i s \neq 0 \vee a_i^T s \leq 0$ for $i = m_1 + 1, \; \dots \; , \; k$, i.e. conditions of Lemma 2.4.1 are satisfied. The backward implication is trivial.

∎

The next result follows directly from Theorems 3.4.1 and 3.4.2 and so no proof is required.

Lemma 4.4. *If $Q(x)$ is unbounded from above on $\mathcal{R}$ along some direction s, then this function is bounded from below in this direction. If function $Q(x)$ is unbounded from below on $\mathcal{R}$ along some direction s then this function is bounded from above in this direction.*

∎

Thus, if the function $Q(x)$ is unbounded from below and above on $\mathcal{R}$, that means that the two kinds of unboundedness occur in different directions. The same statement is, of course, true for the convex region $\mathcal{R}^v$, and the concave region $\mathcal{R}^c$. That Lemma 4.4 holds for indefinite functions $Q(x)$ is shown in the next two lemmas.

Lemma 4.5. *The function $Q(x)$, where the matrix B is indefinite, is unbounded from above in the direction s if and only if*

$$s^T B s > 0 \vee (a^T s > 0 \wedge s^T B s = 0).$$

Proof: Note that $Q(x)$ is unbounded in direction s if and only if it is unbounded along some half-line $x(p, s)$. The result then follows from the expansion $Q(x(p, s))$.

∎

Lemma 4.6. *The function $Q(x)$, where the matrix B is indefinite, is unbounded from below in the direction s if and only if*

$$s^T B s < 0 \vee (a^T s < 0 \wedge s^T B s = 0).$$

Proof: The same as previous Lemma.

∎

Note that now it would be very easy to formulate the necessary and sufficient conditions for unboundedness of the indefinite function $Q(x)$ on $\mathcal{R}^v$ and $\mathcal{R}^c$ by combining Lemma 4.5 and Lemma 4.6 respectively with the conditions for unboundedness of the regions $\mathcal{R}^v$ and $\mathcal{R}^c$. However, checking these conditions is a difficult problem because of the indefiniteness of B. In [15] was shown that problems of this kind, in particular the problem

$$\begin{aligned} &\text{minimize} \;\; x^T Bx \\ &\text{subject to} \;\; x \geq 0, \end{aligned}$$

are NP - hard problems.

Parts 1 and 2 of the next corollary follow immediately from Theorems 3.3.2 and 3.3.1, respectively. They deal with the special case when the constraint Hessian matrices are positive definite.

Lemma 4.7.

(1) *If the matrices B_i, $i \in I_{m\backslash m_1}$ are positive definite, then the function $Q(x)$ is unbounded from above on $\mathcal{R}^c$ if $B \neq 0 \vee a \succeq 0$ (i.e. $a \geq 0$ and $a \neq 0$).*

(2) *If the matrices B_i, $i \in I_{m\backslash m_1}$ are positive definite, then the function $Q(x)$ is unbounded from below on $\mathcal{R}^c$ if and only if there exists a vector s satisfying the conditions*

$$\begin{aligned} Bs &= 0 \\ a^T s &< 0. \end{aligned}$$

(3) *If* $m - m_1 < d + 1$ *then the region* $\mathcal{R}^c$ *is unbounded.*

Proof: The proof of part 3 follows from the well-known fact that if the set $a_i x \leq c_i$, $i \in I_{m \backslash m_1}$ is bounded then $m - m_1 \geq d + 1$.

∎

Lemma 4.8. *If the system* $a_i^T s \leq 0$, $i \in I_{m_1}$, *has only the zero solution and if* $m_1 > 2d$, *then we can choose a subset* $\{i_1, \ldots, i_{m^1}\}$ *of* I_{m_1} *where* $m^1 \leq 2d$, *such that the region defined by the constraints*

$$(1/2) x^T B_{i_j} x + a_{i_j}^T x \leq c_{i_j}, \quad j = 1, ..., m^1,$$

is bounded.

Proof: The region $\mathcal{R}^v$ is bounded, because the region defined by the system $a_i^T x \leq c_i$, $i \in I_{m_1}$ is bounded. The proof then follows from Lemma 1 in [2] which states that if $a_i x \leq 0, i \in I_{m_1}$ is a prime representation of $P = \{0\} \subset \Re^d$, then $d+1 \leq m_1 \leq 2d$. A prime representation is one that contains no redundant constraints or implicit equalities.

∎

The next lemma is an extension of Lemma 2.3.2. It explicitly constructs a half-line $x(p, s)$ along which the quadratic terms are unbounded.

Lemma 4.9. *If, for each* $i \in Z^c$, *the term* $x^T B_i x$ *is unbounded from above along the half-line* $x(p_i, s_i) \in R_Z$, *then all the terms* $x^T B_i x$, $i \in Z^c$, *are unbounded along the half-line* $x(p, s) \in R_Z$, *where*

$$p = (1/|Z^c|)\sum_{i\in Z^c} p_i, \text{ and } \quad s = \sum_{i\in Z^c} \alpha_i s_i$$

and where $|Z^c|$ denotes number of elements of the set Z^c and the α_i are arbitrary nonnegative numbers.

Proof: We will show only that $x(p,s) \in R_Z$, i.e., that $a_i^T\, x(p,s) \le c_i,\ i \in Z^c$. The unboundedness of the terms $x^T B_i x,\ i \in Z^c$, follows directly from Lemma 2.3.2.

Substituting the expression for the half-line $x(p,s)$ where p and s are as given above, into the inequalities $a_i^T x \le c_i,\ i \in Z^c$, gives

$$a_i^T [\sum_{j\in Z^c} p_j/|Z^c| + t\sum_{j\in Z^c} \alpha_j s_j] \le c_i,\ i \in Z^c$$

$$\leftrightarrow (\sum_{j\in Z^c} a_i^T p_j)/|Z^c|) + t\sum_{j\in Z^c} \alpha_j a_i^T s_j \le c_i,\ i \in Z^c.$$

After substituting $t = \tau/|Z^c|$ we have

(4.1) $$(\sum_{j\in Z^c} a_i^T(p_j + \tau\alpha_j))/|Z^c| \le c_i,\ i \in Z^c.$$

Since each half-line $x(p_i, s_i)$ lies in the set R_Z, we have

(4.2) $$a_i^T(p_j + t_j s_j) \le c_i \quad for \quad t_j \ge 0, \quad j \in Z^c, \quad i \in Z.$$

Substituting $t_j = \tau\alpha_j$ into (4.2) and then adding all inequalities in (4.2), we get (4.1).

∎

The reason that we examine the boundedness of the terms $x^T B_i x$, $i \in Z^c$ over R_Z, is that if these terms are bounded, then we could add the index i to the set Z which is constructed by Algorithm A.

In Lemma 4.9 the half-line lies in the region R_Z. The next lemma shows that we can also construct a half-line in $\mathcal{R}^c$.

Lemma 4.10. *If, for each $i \in Z^c$, the term $x^T B_i s$ is unbounded from above along the half-line $x(p_i, s_i) \in R_Z$, then there exists a half-line $x(p, s) \in \mathcal{R}^c$.*

Proof: From Lemma 4.9 it follows that there exists a positive scalar T such that the half-line $x(p, s) \in \mathcal{R}^c$ for all $t \geq T$, where

$$p = (1/|Z^c|) \sum_{i \in Z^c} p_i \quad \text{and} \quad s = \sum_{i \in Z^c} \alpha_i s_i.$$

∎

It is interesting to note the relationship between the results in this thesis and the concept of constraint redundancy (cf. [11]). From Theorem 3.2.1, it is clear $Q(x)$ is bounded from below on $\mathcal{R}^v$ if and only if the constraint $a^T s \geq 0$ is redundant with respect to the constraints (2.2.1), (2.2.2) and (3.1.1). In addition, it follows from Theorem 3.2.2 that the constraint $a^T x \leq 0$ is redundant with respect to (2.2.1) and (2.2.2) and the constraints $\pm b_i^T x \leq 0$, $i \in I_{m_1}$ are redundant with respect to (2.2.1) and (2.2.2) if and only if $Q(x)$ is bounded from above on $\mathcal{R}^v$. Note that if $\pm b_i^T x \leq 0$, $i \in I_{m_1}$ are redundant, then they are implicit equalities (cf. [25]) with respect to (2.2.1) and (2.2.2). Also, note that if $Q(x)$ is bounded from above and below on $\mathcal{R}^v$, then $a^T x \leq 0$ is an implicit equality with respect to (2.2.1), (2.2.2) and (3.1.1).

Clearly, the above remarks apply to the corresponding constraints in the reduced variable LP problems. The corresponding constraints redundancy (implicit equality) problems have the special structure exploited in [3] to develop an efficient method, called DEPS, for handling the redundancy problem in the presence of degeneracy. This algorithm, easily modified to detect implicit equalities, is ideally suited for the problem of determining the boundedness of $Q(x)$ on $\mathcal{R}^v$.

It should be also noted, that conditions for lower boundedness of $Q(x)$ on $\mathcal{R}^v$ can be stated in another equivalent form. Namely since equations (2.2.1), (2.2.2), (3.1.1) and (3.1.2) are equivalent to System I of Slater's theorem of the alternative [12], the following lemma is immediate.

Lemma 4.11. *The function $Q(x)$ is bounded from below on $\mathcal{R}^v$ if and only if there exists scalars u and $u_1, \ldots, u_{m_1}$ and d-vectors v and $v_1, \ldots, v_{m_1}$ satisfying*

$$au + a_1 u_1 + \cdots + a_{m_1} u_{m_1} + Bv + B_1 v_1 + \ldots + B_{m_1} v_{m_1} = 0,$$

where either $u > 0$ and $(u_1, ..., u_{m_1})^T \geq 0$ or $u = 0$ and $(u_1, ..., u_{m_1})^T > 0$.

∎

Finally, we note that if there are linear equality constraints, say $Ax = b$, then it is only necessary for the Hessian matrices to be positive semidefinite on the null space of A, rather than on $\Re^d$.

CHAPTER 5

CONCLUDING REMARKS

We have presented necessary and sufficient conditions for the existence of upper and lower bounds on a convex quadratic function over a feasible regions defined by convex and concave quadratic constraints. We also gave necessary and sufficient conditions for the unboundedness of the feasible region.

In order to determine whether or not the conditions can be satisfied, we made extensive use of Algorithm A. We note that, in fact, Algorithm A is a mechanism for dealing with determining a solution to systems as in (1.1). Also, Algorithm A must solve a finite number of LPs of the form $\min\{a^T s \mid a_i^T s \leq 0,\ i \in I_m\}$. As was mentioned in Chapter 1, the numerical procedures for determining unboundedness require only the solution of a finite number of LPs. Since the LP problem is polynomial, then the problem of determining unboundedness is also polynomial. The maximum number of LPs for each problem is given below.

unboundedness problem	max. number of LPs
$\mathcal{R}^v$	$2d$
$\mathcal{R}^c$	$d(m-m_1-1)(2+m-m_1)$
$\mathcal{R}$	$d(m-m_1-1)(2+m-m_1)$
$Q(x)$ from below on $\mathcal{R}^v$	1
$Q(x)$ from above on $\mathcal{R}^v$	$2d+1$
$Q(x)$ from below on $\mathcal{R}^c$	$d(m-m_1-1)(2+m-m_1)+1$
$Q(x)$ from above on $\mathcal{R}^c$	$d(m-m_1)(3+m-m_1)+1$
$Q(x)$ from below on $\mathcal{R}$	$d(m-m_1-1)(2+m-m_1)+1$
$Q(x)$ from above on $\mathcal{R}$	$d(m-m_1)(3+m-m_1)+1.$

We note that it seems unlikely that it would be necessary to solve the maximal number of LPs, particularly if we take into account the observation made after Algorithm A at the end of Section 2.3. We also note that the maximal number of LPs can be more than halved if there are nonnegativity constraints. This is a direct consequence of the fact that, in this case, Lemma 1.1 can be simplified. That is, it can be shown that $Ps = 0$ for all $s \in R'$ if and only if $q_k = \text{maximum } \{p_k^T s \mid s \in R'\} = 0$, for each $k \in I_d$. This follows from the inequality $0 \leq s^T P s \leq q^T s = 0$, where q is the vector with components q_j. Moreover, in Step 2 of the Algorithm A, instead of performing $2d$ LPs we have only one LP of the form

$$\max\{\sum_{i=1}^{n} x_i | x \in R'_{Z_n}\}.$$

Thus, for example, maximal number of LPs necessary for checking of the boundedness of the set $\mathcal{R}$ with positivity conditions is equal to

$$(m-m_1-1)(1+d(m-m_1)/2).$$

Future research in this area can proceed in many directions. The three most promising are

(1) Determine necessary conditions for unboundedness of $\mathcal{R}$ and necessary conditions for unboundedness of $Q(x)$ from below and above on $\mathcal{R}$.

(2) Determine ways to efficiently solve the LPs that result from Lemma 1.1. Note that these LPs also arise in the solution of degenerate linear programs in that their solution can determine a nondegenerate pivot.

(3) Determine a new algorithm for solving the QPQC problem. The algorithm in [23] requires that the feasible region be bounded by a polytope. Perhaps the results in this thesis can be used to eliminate that requirement.

APPENDIX A

In this appendix we give a numerical procedure which can be used throughout the thesis to use the equality constraints (all having zero right hand sides) to reduce the number of variables in the LPs to be solved.

ALGORITHM C.

Purpose: To find a basis for the intersection of the null spaces of the symmetric positive semidefinite matrices B_i, $i \in I_k$.

Step 0: Set $B_i^* = B_i, \quad \forall i \in I_k$. Set $d_1 = d$ and $n = 1$.

Step 1: Determine a matrix N_n (see below) whose columns form a basis for $N(B_n^*)$. If $N(B_n^*)$ contains only the zero vector then the intersection of the null spaces of $B_1, ..., B_n$ is the zero vector. In this case stop. Otherwise, set $d_{n+1} = rank(N_n)$.

Step 2:
If $n \leq k$, then replace B_i^* with $N_n^T B_i^* N_n$ for $i = n+1, ..., k$.

Step 3: If $B_i^* = 0$ for $i = n+1, ..., k$, then stop. If $n = k$, then stop. If $B_i^* = 0$ for $i = n+1, ..., j$, then replace n with $j+1$. Otherwise replace n with $n+1$. Go to Step 1.

∎

There are two reasonable approaches to the computation of rank(B_n^*) and N_n. The first is to compute the QR decomposition [8]

$$\begin{pmatrix} Q_n^1 \\ Q_n^2 \end{pmatrix} B_n^* = \begin{pmatrix} R_n \\ 0_n \end{pmatrix}.$$

It follows that rank(B_n^*) is equal to the number of rows in R_n, and that we can take $N_n = (Q_n^2)^T$. The second is to compute the Choleski decomposition [8] $P^T(B_n^*)P = L_n L_n^T$, where

$$L_n^T = \begin{pmatrix} R_n & S_i \\ 0 & 0 \end{pmatrix}.$$

In this case rank(B_n^*) is equal to the number of rows in R_n, and we can take

$$N_n = P \begin{bmatrix} -R_n^{-1} S_n \\ I \end{bmatrix}.$$

After the algorithm is completed, provided the intersection of the null spaces is nonempty, we calculate the matrix

$$N = \prod_{i \in I_k} N_i$$

and the change of variables is given by

$$x = Ny.$$

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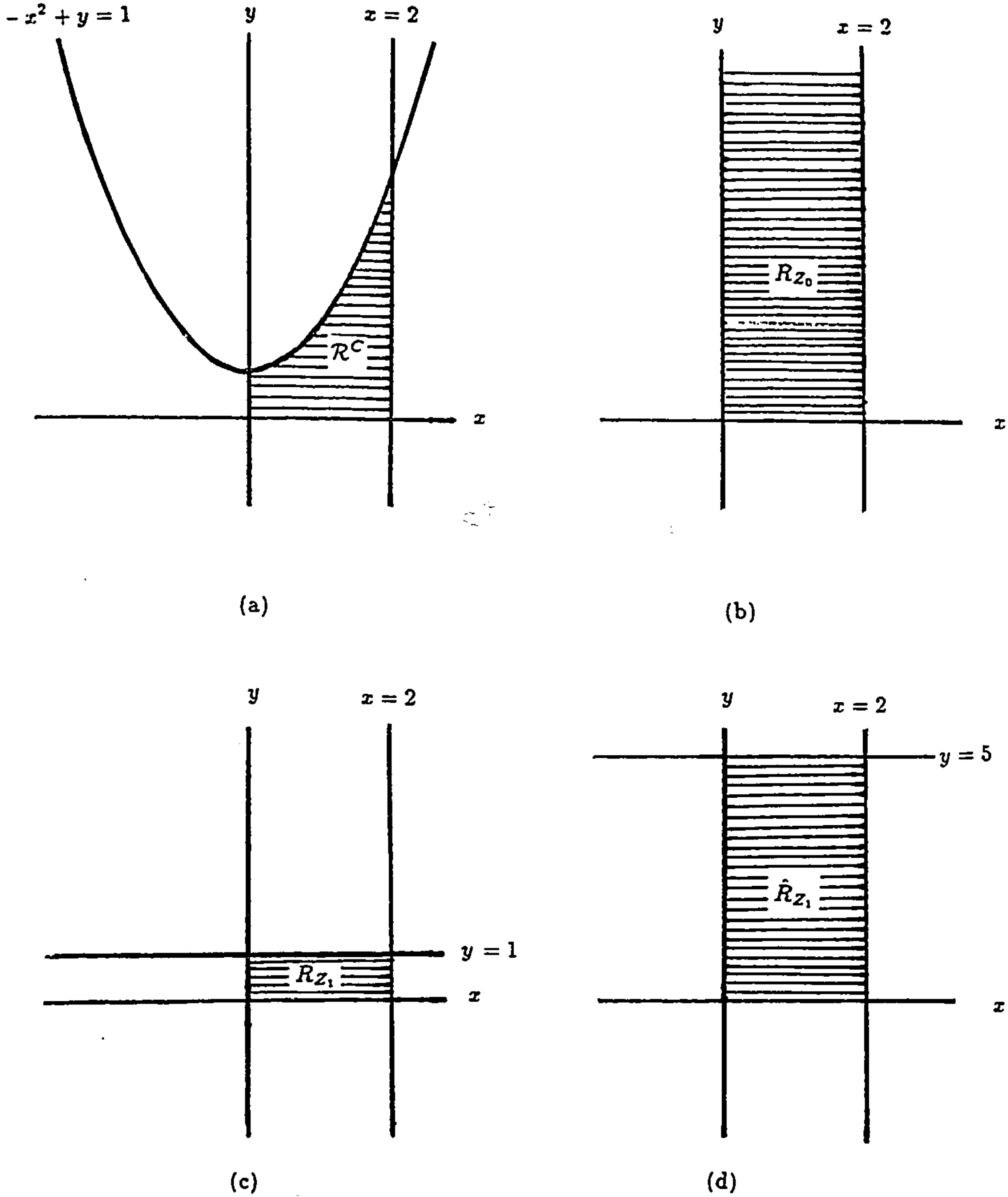

Figure 1: Graphs for Example 1.

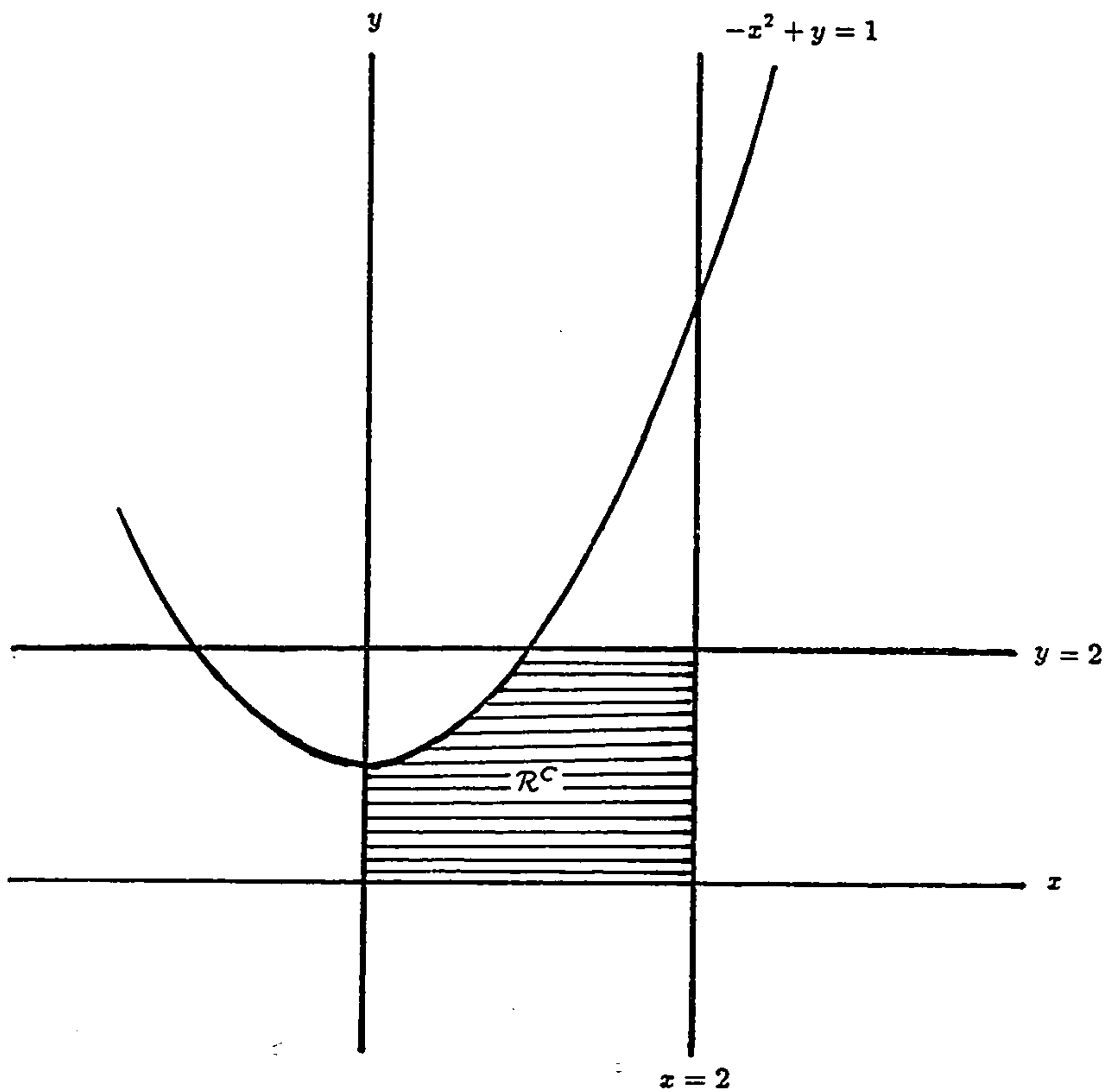

Figure 2: Graph of $\mathcal{R}^c$ for Example 2.

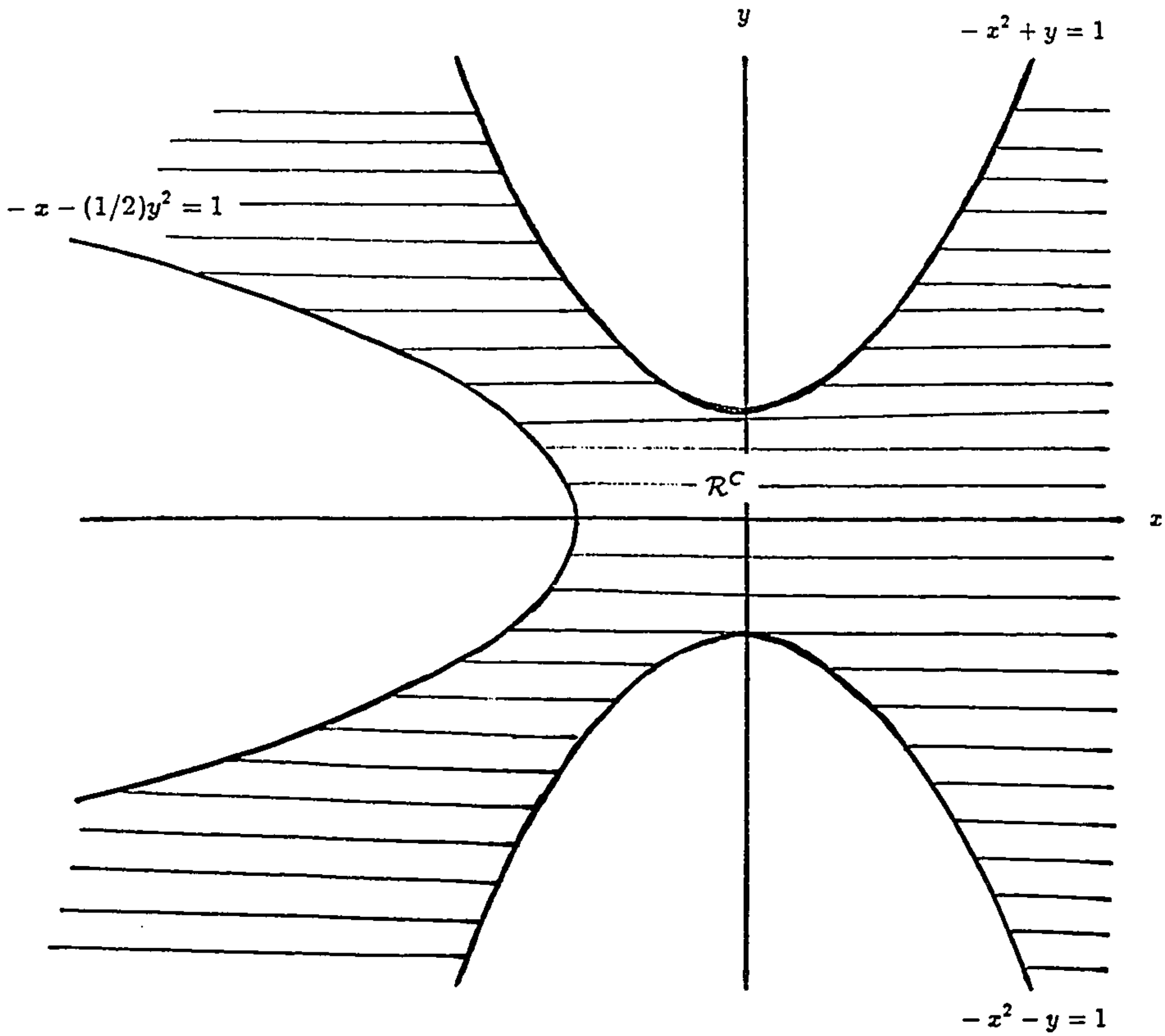

Figure 3: Graph of $\mathcal{R}^c$ for Example 3.

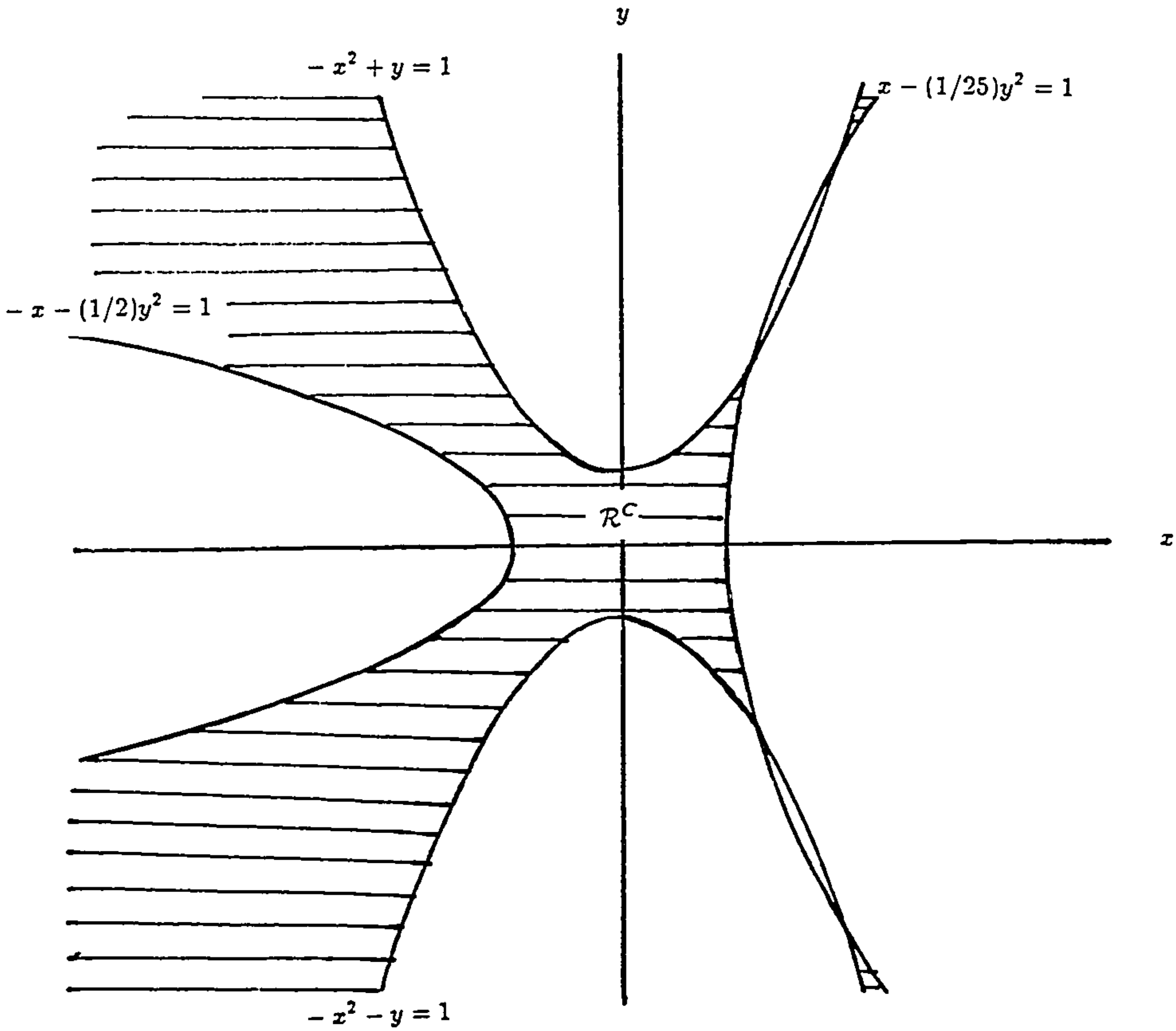

Figure 4: Graph of $\mathcal{R}^c$ for Example 4.

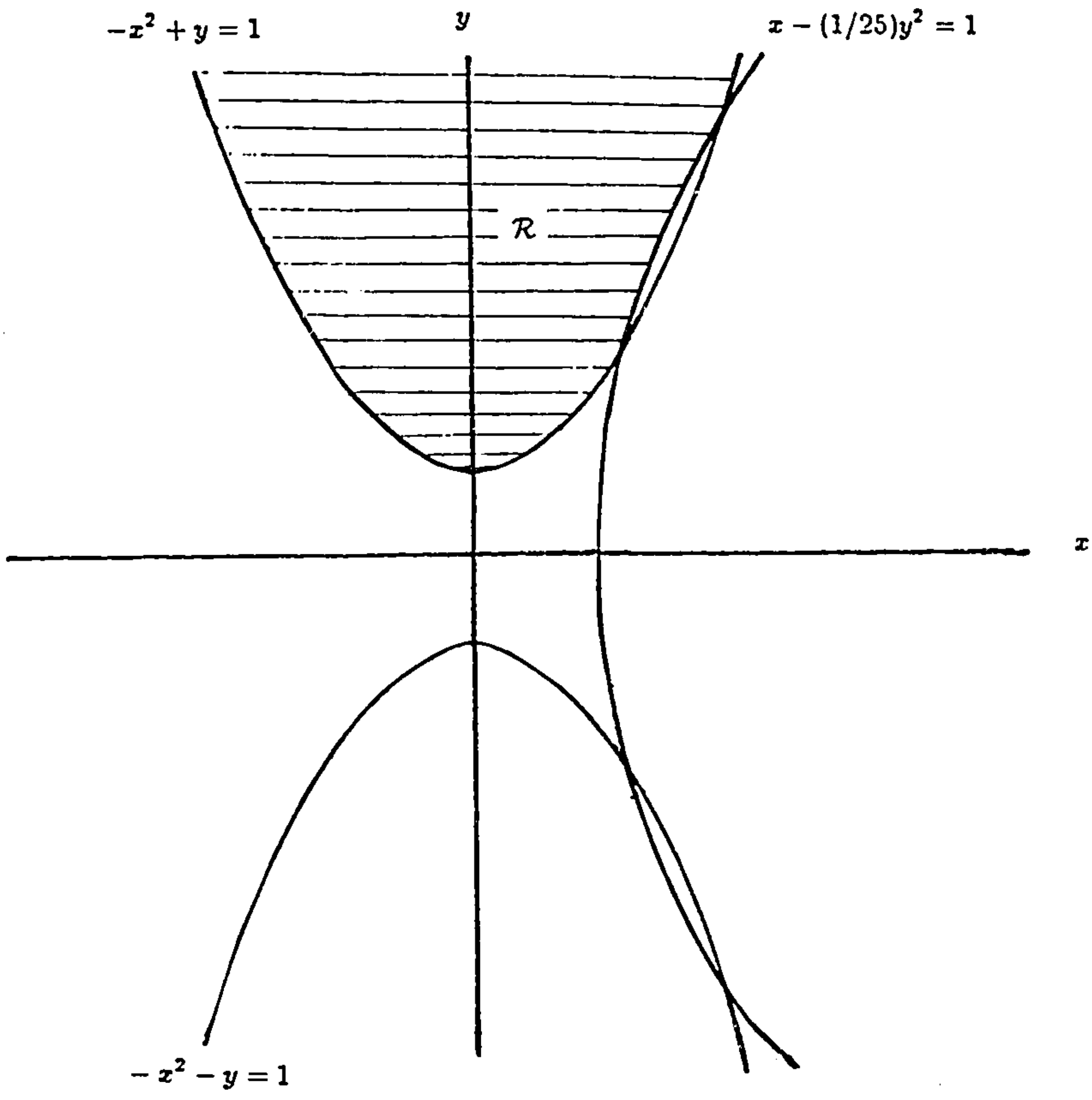

Figure 5: The region $\mathcal{R}$ for Example 5.

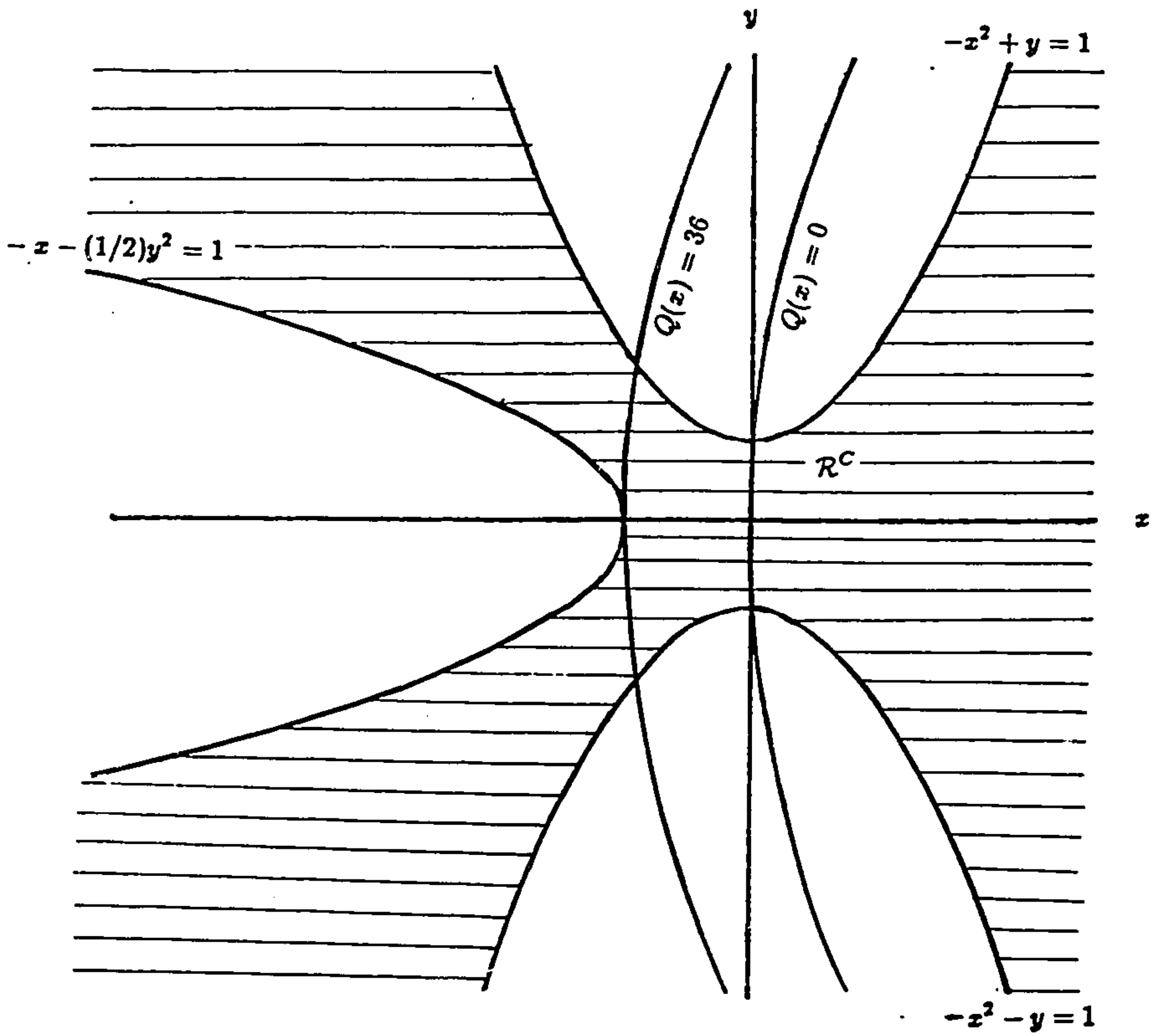

Figure 6: The graph for Example 10. The function $Q(x)$ is unbounded from below and above.

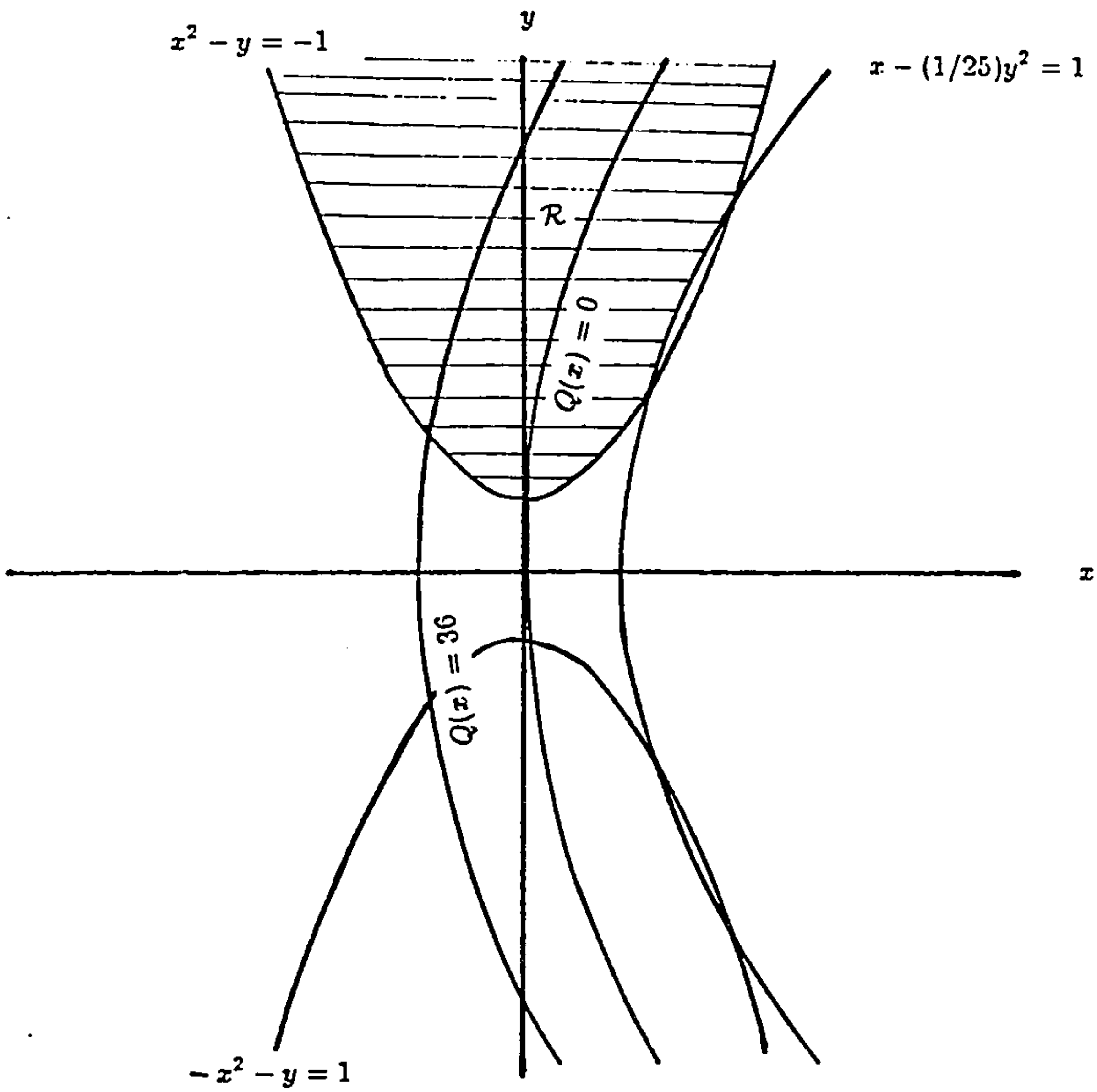

Figure 7: The graph for Example 11. The function $Q(x)$ is bounded from below and unbounded from above.

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